GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Electronics and Communication Engineering

HANDOUT

on

NUMERICAL METHODS & COMPLEX ANALYSIS

Vision

To be a leading centre of education and research in Electronics and Communication Engineering, making the students adaptable to changing technological and societal needs in a holistic learning environment.

Mission

- To produce knowledgeable and technologically competent engineers for providing services to the society.
- To have a collaboration with leading academic, industrial and research organizations for promoting research activities among faculty and students.
- To create an integrated learning environment for sustained growth in electronics and communication engineering and related areas.

Program Educational Objectives

Graduates of the Electronics and Communication Engineering program will

1. demonstrate a progression in technical competence and leadership in the practice/field of engineering with professional ethics.
2. continue to learn and adapt to evolving technologies for catering to the needs of the society.

HANDOUT ON NUMERICAL METHODS & COMPLEX ANALYSIS

Class & Sem.:I B.Tech – II Semester Year: 2018-19
Branch: ECE Credits:3

==

1. **Brief History and Scope of the Subject**

   "MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge, which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

   <u>Scope of mathematics in engineering study :</u>

   Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

   The most important objective and purpose in Engineering Mathematics is that the students becomes familiar with Mathematical thinking and recognize the guiding principles and ideas "Behind the science" which are more important than formal manipulations. The student should soon convince himself of the necessity for applying mathematical procedures to engineering problems.

2. **Pre-Requisites**

   Basic Knowledge of Mathematics such as differentiation and Integration at Intermediate Level is necessary.

3. **Course Objectives:**
   - To understand the various numerical techniques.
   - To introduce the complex functions, complex differentiation and complex integration.
   - To introduce the concepts of conformal and bi linear transformations of standard functions.

4. **Course Outcomes:**

   Students will be able to

   CO1: Apply numerical techniques for solutions of Algebraic, transcendental and ordinary differential equations.
   CO2: compute interpolating polynomial for the given data.
   CO3: find derivatives and integrals by using numerical techniques.
   CO4: test the differentiability (analyticity) of a complex function.
   CO5: Find complex integration with the use of Cauchy's integral formula.
   CO6: Apply the concepts of conformal and bilinear transformations of standard functions.

5. **Program Outcomes:**

   The graduates of electronics and communication engineering program will be able to

   a) Apply knowledge of mathematics, science, and engineering for solving intricate engineering problems.
   b) Identify, formulate and analyze multifaceted engineering problems.
   c) Design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability.
   d) Design and conduct experiments based on complex engineering problems, as well as to analyze and interpret data.
   e) Use the techniques, skills and modern engineering tools necessary for engineering practice.
   f) Understand the impact of engineering solutions in a global, economic and societal context.
   g) Design and develop eco-friendly systems, making optimal utilization of available natural resources.

h) Understand professional ethics and responsibilities.
i) Work as a member and leader in a team in multidisciplinary environment.
j) Communicate effectively.
k) Manage the projects keeping in view the economical and societal considerations.
l) Recognize the need for adapting to technological changes and engage in life-long learning.

6. Mapping of Course Outcomes with Program Outcomes:

	a	b	c	d	e	f	g	h	i	j	k	l
CO1	**H**	**M**										
CO2	**M**	**L**										
CO3	**M**											
CO4	**H**	**M**										
CO5				**H**								
CO6				**M**								

7. Prescribed Text Books

1. B.S.Grewal, Higher Engineering Mathematics : 42nd edition, Khanna Publishers,2012 , New Delhi.
2. B.V Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill Company Ltd.

8. Reference Text Books

1. U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2nd Edition, Excel Books, 2011, New Delhi.
2. Erwin Kreyszig, Advanced Engineering Mathematics: 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.

9. URLs and Other E-Learning Resources

Sonet CDs & IIT CDs on some of the topics are available in the digital library.

10. Digital Learning Materials:

- **http://nptel.ac.in/courses/106106094**
- **http://nptel.ac.in/courses/106106094/40**
- **http://nptel.ac.in/courses/106106094/30**
- **http://nptel.ac.in/courses/106106094/32**
- **http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf**

11. Lecture Schedule / Lesson Plan

Topic	**No. of Periods**	
	Theory	**Tutorial**
UNIT –I : Algebraic and Transcendental Equations		
Indroduction	**1**	**1**
Bisection Method	**2**	
Method of False Position	**2**	
Newton – Raphson Method	**2**	
UNIT-II : Interpolation		
Introduction	**1**	**1**
Difference Operators and their relations	**3**	
Forward and Backward difference tables	**2**	**1**
Newton's forward and backward formulae's	**2**	
Lagrange's interpolation	**2**	
UNIT-III: Numerical Differentiation and Integration		
Introduction	**1**	**1**
Newton's forward and backward formulas for first and second derivatives	**3**	

Trapezoidal formulas for integrations	**2**	**1**
Simpson's formulas for integrations	**2**	
UNIT-IV: Numerical Solution of Ordinary Differential Equations		
Introduction	**1**	**1**
Taylor's Method	**3**	
Euler's and Euler's Modified Method	**3**	
Runge – Kutta Method	**3**	**1**
UNIT-V: Complex Differentiation & Integration		
Introduction	**1**	**1**
Continuous and derivatives of Complex functions	**2**	
C-R equations and their applications	**4**	**1**
Line integral and Cauchy's theorem	**3**	
Cauchy's integral formulae	**3**	
UNIT-VI:Conformal Mapping		
Introduction	**1**	**1**
Tranformations of e^z, sinz, cosz, z+1/z, lnz	**4**	
Fixed points, cross ratio points	**1**	**1**
Bilinear and inverse transformations	**2**	
Total No. of Periods:	**56**	12

12. Seminar Topics

- Convergence of Bisection, Regula –Falsi and Newton-Raphson Methods
- Modeling tabulated data for estimates the quantity 'y' at any value of controllable value 'x' for physical and chemical experiments.
- Finding contour integrations along the paths.

UNIT-I

Algebraic and Transcendental equations

Objectives:

- To introduce various numerical methods for solving algebraic and transcendental equations.

Syllabus: Solution of Algebraic and Transcendental equations-Introduction-Bisection method-Method of false position-Newton-Raphson method.

Course outcomes:

After completion of the unit, students will able to

- Solve an algebraic or transcendental equation using an appropriate numerical method **.**

Introduction:

A problem of great importance in science and engineering is that of determining the roots/ zeros of an equation of the form f(x) = 0.

Polynomial function: A function f(x) is said to be a polynomial function if f(x) is a polynomial in x. i.e. f(x) $= a_0 x^n + a_1 x^{n-1} + \dots\dots\dots\dots + a^{n-1} x + a_n$ where $a_0 \neq 0$, the coefficients $a_0, a_1 \dots\dots\dots\dots a_n$ are real constants and n is a non-negative integer.

Algebraic function: A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = c_1 e^x + c_2 e^{-x}$

$$f(x) = e^{5x} - \frac{x^3}{2} + 3$$

Algebraic Equation: If f(x) is an algebraic function, then the equation f(x) =0 is called an algebraic equation.

Transcendental Equation: An equation which contains polynomials, exponential functions, logarithmic functions and Trigonometric functions etc. is called a Transcendental equation.

Ex:- $xe^{2x} - 1 = 0$, $\cos x - x e^x = 0$, $\tan x = x$ are transcendental equations.

Root of an equation: A number α is called a root of an equation f(x) = 0 if f(α) =0. We also say that α is a zero of the function.

Note:

- The roots of an equation are the abscissas of the points where the graph y = f(x) cuts the x-axis.
- A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or infinite number of roots depending on the form of f(x).

Direct method:

We know the solution of the polynomial equations such as linear equation $ax+b=0$ and quadratic equation $ax^2+bx+c=0$, will be obtained using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also well known to us. There are no direct methods for solving higher degree algebraic equations or equations involving transcendental functions, such equations are solved by numerical methods. In these methods we find a interval in which the root lies. We use the following theorem of calculus to determine an initial approximation. It is also called the Intermediate value theorem.

Intermediate value theorem: If f(x)is continuous on some interval [a, b]and f(a)f(b)< 0, then the equation f(x) = 0 has at least one real root in the interval (a, b). In this unit we will study some important methods of solving algebraic and transcendental equations.

Bisection method: Bisection method is a simple iteration method to solve an equation. This method is also known as" Bolzano method of successive bisection" and it referred as" Half-interval method". Suppose we know an equation of the form $f(x)=0$ has exactly one real root between two real numbers x_0, x_1.The number is chosen such that $f(x_0)$ and $f(x_1)$will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the midpoint $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2)=0$ then x_2 is a root. If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2. The interval is taken as (x_0, x_2) Otherwise the root lies in the interval $[x_2, x_1]$. Repeating this process until we obtained successive subintervals which are smaller.

Problems:

1.Find a root of the equation $x^3-5x+1=0$ using the bisection method in 5 – stages

Sol: Let $f(x)=x^3-5x+1$, we note that f(0) > 0 and f(1)< 0

$\therefore$ Root lies between 0 and 1

Consider $x_0=0$ *and* $x_1=1$

By bisection method the next approximation is

$$x_2 = \frac{x_0+x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ *and* $f(0) > 0$

Since $f(0)>0$, we conclude that root lies between x_0 *and* x_3

The third approximation of the root is

$$x_4 = \frac{x_0+x_3}{4} = \frac{1}{2}(0+0.25)$$

$$= 0.125$$

We have $f(x_4)=0.37495>0$

Since $f(x_4)>0$ *and* $f(x_3)<0$, the root lies between

$x_4=0.125$ *and* $x_3=0.25$

Considering the 4th approximation of the roots

$$x_5=\frac{x_3+x_4}{2}=\frac{1}{2}(0.125+0.25)=0.1875$$

$f(x_5)=0.06910>0$,

since $f(x_5)>0$ *and* $f(x_3)<0$ the root must lie between $x_5=0.18758$ and $x_3=0.25$

Here the fifth approximation of the root is

$$x_6=\frac{1}{2}(x_5+x_3)$$

$$=\frac{1}{2}(0.1875+0.25)$$

$$=0.21875$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it between 0 and 1.

False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x)=0$. Consider two initial approximate values x_0 *and* x_1 near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between x_0 *and* x_1. The curve $f(x)$ crosses x-axis only once at the point x_2 lying between the points x_0 *and* x_1, Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line, Suppose this line cuts x-axis at x_2, We calculate the values of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite sign, then the root lies between x_0 *and* x_2 and value x_1 is replaced by x_2

Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2

Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points x_2, x_3, x_4 ..... Obtained converge to the expected root of the equation $y=f(x)$.

To obtain the equation to find the next approximation to the root

Let $A=\left(x_0, f\left(x_0\right)\right)$ *and* $B=\left(x_1, f\left(x_1\right)\right)$ be the points on the curve $y=f\left(x\right)$ Then the equation to the chord AB is $\frac{y-f\left(x_0\right)}{x-x_0}=\frac{f\left(x_1\right)-f\left(x_0\right)}{x-x_0}\rightarrow(1)$

At the point C where the line AB crosses the x – axis, we have f(x) =0 i.e. y =0

From (1), we get $x=x_0-\frac{x_1-x_0}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right)\rightarrow(2)$

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new values of x is taken as x_2 then (2) becomes

$$x_2=x_0-\frac{\left(x_1-x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right)$$

$$=\frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)}. \text{ ---------------3}$$

Now we decide whether the root lies between

x_0 *and* x_2 $(or) x_2$ *and* x_1

We name that interval as $\left(x_1, x_2\right)$ The line joining $\left(x_1, y_1\right)\left(x_2, y_2\right)$ meets x – axis at x_3 is given by $x_3=\frac{x_1 f\left(x_2\right)-x_2 f\left(x_1\right)}{f\left(x_2\right)-f\left(x_1\right)}$

This will in general, be nearest to the exact root we continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as $\left(x_0, x_1\right),\left(x_1, x_2\right),\left(x_2, x_3\right)$ etc

Where $x_i<x_i+1$ and $f\left(x_0\right), f\left(x_i+1\right)$ are of opposite signs

Also $x_{i+1}=\frac{x_{i-1} f\left(x_i\right)-x_i f\left(x_{i-1}\right)}{f\left(x_i\right)-f\left(x_{i-1}\right)}$

Problems:-

2. Find out the roots of the equation $x^3-x-4=0$ using false position method

Sol: Let $f(x) = x^3 - x - 4 = 0$. By verification, $f(0) = -4$, $f(1) = -4$ and $f(2) = 2$. Since $f(1)$ and $f(2)$ have opposite signs the root lies between 1 and 2.

By false position method, $x_2=\frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)}$

$$x_2=\frac{(1\times 2)-2(-4)}{2-(-4)}$$

$$=\frac{2+8}{6}=\frac{10}{6}=1.666$$

$$f(1.666)=(1.666)^3-1.666-4$$

$$=-1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$

The root is 1.796

Newton- Raphson Method:

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1) = 0$.

By Taylor's theorem neglecting second and higher order terms $f(x_1) = f(x_0+h) = 0 \Rightarrow f(x_0) + h f'(x_0) = 0$.

$\therefore$ $h = - f(x_0)/f'(x_0)$

Substituting this in x_1 we get

$$x_1 = x_0 + h$$
$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

Problem:-

3.Find by Newton's method, the real root of the equation $xe^x - 2 = 0$ Correct to three decimal places.

Sol. Let $f(x) = xe^x - 2 \rightarrow (1)$

Then $f(0) = -2$ and $f(1) = e - 2 = 0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. so we take $x_0 = 1$ and $f^1(x) = xe^x + e^x$ *and* $f^1(1) = e + e = 5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f^1(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \dfrac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$

$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

Convergence of the Iteration Methods

We now study the rate at which the iteration methods converge to the exact root, if the initial approximation is sufficiently close to the desired root.

Define the error of approximation at the k th iterate as $\epsilon_k = x_k - \alpha$, k = 0, 1, 2,

Definition: An iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant C≠0 , such that

$$|\epsilon_{k+1}| < | \epsilon_k^p |$$

The constant C, which is independent of k, is called the asymptotic error constant and it depends on the derivatives of f(x) at x = α.

SECTION-A

Assignment-cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective Questions

1. The formula to find *(n +1)th* approximation of root of *f(x) = 0* by Newton Raphson method is

a) $x_{n+1} = x_n - \dfrac{f(x)}{f(x_{n+1})}$ b) $x_{n+1} = x_n - \dfrac{f^1(x_n)}{f(x_n)}$

c) $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$ d) $x_{n+1} = x_n \frac{f(x_n)}{f^1(x_n)}$

2. Every algebraic equation of nth degree has exactly --------- roots.
3. Which of the following alter name for method of falsi method is
(a) Method of chords (b) Method of tangents (c) Method of bisection (d) Regula-falsi method
4. In bisection method if root lies between a and b then *f(a).f(b)* --------------
5. Under the conditions that f(a) and f(b) have apposite signs and *a*<*b*, the first approximation of one of the roots f(x)=0 by Regula-falsi method is given by
a) $x_1 = \frac{af(a)-bf(b)}{f(a)-f(b)}$ b) $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$
c) $x_1 = \frac{af(a)+bf(b)}{f(a)+f(b)}$ d) $x_1 = \frac{af(b)+bf(a)}{f(b)+f(a)}$
6. Write an example of a Transcendental equation ___________________________________
7. The interval in which the root of $f(x) = x^3 -2x-5 =0$ lies
a) (3,5) b) (1,2) c) (2,3) d) (5,1)
8. A root of $x^3 -5x^2+7 = 0$ lies between 4 and 5 (true or false)
9. If $f(0)$ and $f(1)$ are of opposite signs for $f(x) = \cos x-3x+1$ then root lies between ___________
10. The order of convergence in Newton- Raphson method is
a) 2 b) 3 c) 0 d) 1
11. Newton -Raphson method fails when -----------------------------
12. If x_0 *and* x_1 are 1.4 and 1.5 by false position method find x_2 for $x^2-1-\sin x=0$
a) 1.0009 b) 1.2097 c) 1.1940 d) 1.4091
13. If first two approximations x_0 *and* x_1 of root of $x^3-x^2-2=0$ are 1.5 and 2 then x_2 by regula falsi method is
a) 1.652 b) 1.724 c) 1.892 d) 1.928
14. If first two approximations of root of $xe^x-3=0$ are 1 and 1.5 then x_2 by regula falsi method is
a) 1.21 b) 1.425 c) 1.035 d) 1.312
15. If first two approximations x_0 *and* x_1 for the root of $x^3-3x-4=0$ are 2.125 and -3 then x_2 by Regula- falsi method is
a) -2.521 b) -2.34 c) -2.171 d) -2.79
16. Newton's iterative formula to find the value of $3\sqrt{N}$ is
a) $x_{n+1} = \frac{1}{3}\left(2x_n - N/x_n^2\right)$ b) $x_{n+1} = \frac{1}{3}\left(2x_n + N/x_n^2\right)$
c) $x_{n+1} = \frac{1}{3}\left(x_n - N/x_n^2\right)$ d) $x_{n+1} = \left(2x_n - N/x_n^2\right)$

17. If first approximation x_0 for the root at $x\log_{10}x-1.2=0$ is 2.74 then x_1 by Newton Raphson method is
    a) 2.741 b)2.73 c) 2.751 d) 2.82
18. If first approximation of roots $x^2 - 3x - 5 = 0$ is $x_0 = 4$ then x_1 by Newton Raphson method is
    a)4.20569 b) 4.20 c) 4.20833 d) 4.199836
19. If $f(x)=x^4-4x-10, x_1=$ first approximation of root is 1.858 then x_2 by Newton Raphson method is
    a) 1.861 b) 1.872 c) 1.855 d) 1.92
20. If the length of the interval to begin with is given to be , find out the minimum number of iterations required to be carried
    out to achieve an accuracy of
    a) 0.001 b) 0.0001 c) 0.00001 d) 0.01

SECTION-B

Descriptive Questions

1. Derive the formula for Newton-Raphson Method .
2. Write shot notes on Bisection method.
3. Explain the procedure involved in finding the solution by Regula-Falsi method.
4. Find a positive real root of f(x) = cosx + 1 - 3x = 0 correct to two decimal places by bisection method.
5. Using Newton- Raphson method find a root of tanx=1.5 x which is near x=1.5.
6. Find out the roots of the equation $x^3 - x - 4 = 0$ by False position method.
7. Find the positive root of the equation $3x^4-2x^2+7x-8 = 0$ by the method of interval halving for [0,1.5]
8. Find a root of the equation: $x^2 - 4x - 9 = 0$ using the Bisection method correct to three decimal places.
9. Find an approximate root of $x\log_{10}^{x}-1.2=0$ by Regula- falsi method
10. Find a positive root of the equation 3x=cosx+1 by Newton-Raphson Method
11. Find a real root of $xe^x-\cos x=0$ by Newton-- Raphson method
12. Find a real root of $e^x\sin x=1$ by Regula-falsi method
13. Using Bisection method, find the negative root of $\mathbf{x^3}$-4x+9 =0
14. By using the bisection method, find a approximate root of the equation $\sin x=\frac{1}{x}$ that lies between x=1 and x=1.5 (measured in radians) carry out computations up to the 7^{th} stage.
15. Find by Newton's method, a root of the equation $3x^3-9x^2+8=0$ lying between 1 and 2 correct to three decimal places
16. Using Newton- Raphson method find a root of 3sinx-2x+5=0 near 3

17. Using Newton – Raphson method
    a) Find square root and cube root of a number N
    b) Find reciprocal of a number
18. Differentiate between the Bisection and the Regula Falsi Method.

SECTION-C

1. The root of x3 - 2x - 5 = 0 correct to three decimal places by using Newton-Raphson method is **(GATE-2015)**
   (a) 2.0946 (b) 1.0404 (c) 1.7321 (d) 0.7011
   Ans: (a)
1. Newton-Raphson method of solution of numerical equation is not preferred when**(GATE-2014)**
   (a) Graph of f(x) is vertical (b) Graph of f(y) is not parallel
   (c) The graph of f(x) is nearly horizontal-where it crosses the x-axis. (d) None
   Ans: (c)

LEARNING RESOURCES:

1. B.S. Grewal, Numerical Methods in engineering and science, Khanna Publishers, 43rd edition, 2014.
2. Dr. M.K. Venkataraman, Numerical Methods in Science and Engineering, National Publishing Co., 2005.
3. V. Ravindranadh, P. Vijayalakshmi, A text book on Mathematical methods, Himalaya publishing house, 2011.

REFERENCE BOOKS/OTHER READING MATERIAL:

1. S.S. Sastry, Introductory Methods of Numerical Analysis, 4th edition, 2005.
2. E. Balagurusamy, Computer Oriented Statistical and Numerical Methods – Tata McGraw Hill., 2000.
3. M.K.Jain, SRK Iyengar and R.L.Jain, Numerical Methods for Scientific and Engineering Computation, Wiley Eastern Ltd., 4th edition, 2003.
4. S. Arumugam, A. Thangapandi Isaac, A. Somasundaram, Numerical methods , 2nd Edition, Scitech publications, 2005.
5. T.K.V. Iyenger, B. Krishna garndhi, S. Ranganadham, M.V.S.S.N. Prasad, Mathematical methods, Revised Edition, 2012.

-----------------@-@-@-@-@---------------

UNIT-II
INTERPOLATION

Objectives:

- To determine unknown functional values for the given data in modern scientific computing.
- To gain the knowledge of Interpolation.

Syllabus:

Interpolation-Introduction, Finite Differences, Forward Differences, Backward Differences, Central Differences, Symbolic relations, Newton's formulae for Interpolation, Lagrange's Interpolation.

Sub Outcomes:

After completion of the unit, Students will able to

- Compute interpolating polynomial for the given data.
  - Apply Lagrange's Interpolation method to approximate functions

Introduction:

Consider the function y = f(x), $x_0<x<x_n$, we understand that we can find the value of y, corresponding to every value of x in the range $x_0<x<x_n$. If the function f(x) is single valued and continuous and is known explicitly then the values of f(x) for certain values of x like $x_0, x_1, \ldots\ldots x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$x : x_0 \quad x_1 \quad x_2 \ldots\ldots\ldots x_n$$

$$y : y_0 \quad y_1 \quad y_2 \ldots\ldots\ldots y_n$$

Satisfying the relation y = f(x) and the explicit definition of f(x) is not known, is it possible to find a simple function say f(x) such that f(x) and $\phi(x)$ agree at the set of tabulated points. This process of finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Finite Differences:-

Here we introduce forward, backward and central differences of a function y = f(x). These differences play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics.

Forward Differences:-

Consider a function y = f(x) of an independent variable x. let $y_0, y_1, y_2 \ldots\ldots y_r$ be the values of y corresponding to the values $x_0, x_1, x_2 \ldots\ldots x_r$ of x respectively. Then the differences $y_1 - y_0$, $y_2 - y_1 \ldots\ldots\ldots\ldots$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, \ldots\ldots\ldots$ that is

$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2.........$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2-----$

Here the symbol Δ is called the forward difference operator

The second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1......$ that is

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the n[th] forward differences are defined by the formula.

$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$

The symbol Δ^n is referred as the n[th] forward difference operator.

Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First order differences	Second order differences	Third order differences	Fourth order differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
x_4	y_4	$\Delta y_3 = y_4 - y_3$			

Backward Differences:-

Let $y_0, y_1......y_r......$ be the values of a function y = f(x) corresponding to the values $x_0, x_1, x_2.............x_r.......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2,....$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1,2,3......... \rightarrow (1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2....... \rightarrow (2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 --- \nabla^2{}_r ----$ i.e.,..

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2$ ....

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3..... \rightarrow (3)$ similarly, the n[th] backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1..... \rightarrow (4)$

If $y = f(x)$ is a constant function , then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the n^th backward difference operator

Backward Difference Table:-

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

Central Differences:-

With $y_0, y_1, y_2 \dots y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2 \dots\dots x_r \dots$ *of* x, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator

Comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2 \dots\dots$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0, 1, 2 \dots\dots \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2 \dots$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{\ 3/2} \dots\dots\dots$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$$

Higher order central differences are similarly defined. In general the n^th central differences are given by

for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1, 2 \dots . \rightarrow (4)$

for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1, 2 \dots . \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0 \ \ for \ \ all \ \ n \geq 1$

Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Symbolic Relations:

E-operator: The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

$E^n y^r = y_{r+n}$ Generalizing

Averaging operator:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$

Relationship Between Δ and E

We have

$$\Delta y_0 = y_1 - y_0$$

$$= Ey_0 - y_0 = (E-1)\,y_0$$

$$\Rightarrow \Delta = E - y(or)E = 1 + \Delta$$

Some more relations

$$\Delta^3 y_0 = (E-1)^3\, y_0 = \left(E^3 - 3E^2 + 3E - 1\right) y_0$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

Inverse operator: Inverse operator E^{-1} defined as $E^{-1} y_r = y_{r-1}$

In general $E^{-n} y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$ ii) $\delta \equiv E^{1/2} - E^{-1/2}$ iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$ v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Differential operator:

The operator D is defined as $Dy(x) = \frac{d}{dx}\left(y(x)\right)$

Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2!}y''(x) + \ldots\ldots\ldots$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x = e^{hD}.y_x$$

We obtain in the relation, $E = e^{hD}$……………………………(3)

•**Theorem:** If *f(x)* is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Note:-

As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1}f(x) = 0$, $\Delta^{n+2} f(x) = 0$,……….

The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function *f(x)* is a polynomial of degree n

1.Find the missing term in the following data

X	0	1	2	3	4
Y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

2. Evaluate

$(i)\,\Delta\cos x$

$(ii)\,\Delta^2\sin(px+q)$

$(iii)\,\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\ \Delta \cos x = \cos(x+h) - \cos x$$
$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\text{h}}{2}$$
$$(ii)\ \Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$$
$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$
$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$
$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$
$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\ \Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$
$$= e^{(ax+b)}\left(e^{ah-1}\right)$$
$$\Delta^2 e^{ax+b} = \Delta\left[\Delta\left(e^{ax+b}\right)\right] - \Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$
$$= \left(e^{ah}-1\right)^2 \Delta\left(e^{ax+h}\right)$$
$$= \left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

Newton's Forward Interpolation Formula:-

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) + ---$$
$$+ b_n(x-x_0)(x-x_1)----(x-x_{n-1}) \rightarrow (1)$$

This polynomial passes through all the points (for i = 0 to n. Therefore, we can obtain the $y_i\text{'}s$ by substituting the corresponding $x_i\text{'}s$ as

$$at\ \ x = x_0, y_0 = b_0$$
$$at\ \ x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$
$$at\ \ x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \rightarrow (1)$$

Let 'h' be the length of interval such that $x_i\text{'}s$ represent

$$x_0, x_0 + h, x_0 + 2h, x_0 + 3h ---- x_0 + xh$$

This implies $x_1 - x_0 = h, x_2 - x_0 - 2h, x_3 - x_0 = 3h ---- x_n - x_0 = nh \rightarrow (2)$

From (1) and (2), we get

$$y_0 = b_0$$
$$y_1 = b_0 + b_1 h$$
$$y_2 = b_0 + b_1 2h + b_2 (2h) h$$
$$y_3 = b_0 + b_1 3h + b_2 (3h)(2h) + b_3 (3h)(2h) h$$

....

....

$$y_n = b_0 + b_1 (nh) + b_2 (nh)(n-1)h + ---- + b_n (nh)\left[(n-1)h\right]\left[(n-2)h\right] \rightarrow (3)$$

Solving the above equations for $b_0, b_{11}, b_2 .....b_n$, we get $b_0 = y_0$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2! h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3! h^3}, b_4 = \frac{\Delta^4 y_0}{4! h^4} ------ b_n = \frac{\Delta^n y_0}{n! h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2! h^2}(x - x_0)(x - x_1)$$
$$+ \frac{\Delta^3 y_0}{3! h^3}(x - x_0)(x - x_1)(x - x_2) + ---- +$$
$$+ \frac{\Delta^n y_0}{n! h^n}(x - x_0)(x - x_1) --- (x - x_{n-1}) \rightarrow (3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph,\ where\ p = 0,1,2,.....n$

Then

$$x - x_1 = x - (x_0 + h) = (x - x_0) - h$$
$$= ph - h = (p-1)h$$
$$x - x_2 = x - (x_1 + h) = (x - x_1) - h$$
$$= (p-1)h - h = (p-2)h$$

....

$$x - x_i = (p - i)h$$

....

$$x - x_{n-1} = \left[p - (n-1)\right]h$$

Equation (3) becomes

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + - - - - - +$$
$$\frac{p(p-1)(p-2) - - - - (p-(n-1))}{n!}\Delta^n y_0 \to (4)$$

Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + - - - - (x - x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_n - 1, ...x_2, x_1, x_0$

We obtain

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2i}\nabla^2 y_n + - - -$$
$$\frac{p(p+1) - - - - [p + (n-1)]}{n!}\nabla^n y_n + - - - - \to (6)$$

Where $p = \frac{x - x_n}{h}$

This uses tabular values of the left of y_n. Thus this formula is useful formula is useful for interpolation near the end of the tabular values

Q:-1. Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q^\circ c)$	205	225	248	274

Sol. The difference table is

x	Y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature be *f(x)*

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54 \ (or)\ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{n!}\Delta^3 y_0+---$$

$$f(54)=205+0.4(20)+\frac{0.4(0.4-1)}{2!}(3)+\frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$

$$=205+8-0.36$$

$$=212.64$$

Melting point = 212.6

Lagrange's Interpolation Formula:-

Let x_1, x_2, x_3, x_4,.......,x_n be the n+1 values of x which are not necessarily equally spaced. Let y_0, y_1, y_2,..........,y_n be the corresponding values of *y= f(x)* let the polynomial of degree n for the function *y= f(x* passing through the (n+1) points $(x_0,f(x_0)),(x_1,f(x_1))----(x_n,f(x_n))$ be in the following form

$$y=f(x)=a_0(x-x_1)(x-x_2)......(x-x_n)+a_1(x-x_0)(x-x_2).........(x-x_n)+$$
$$a_2(x-x_0)(x-x_1).......(x-x_n)+........+a_n(x-x_0)(x-x_1)......(x-x_{n-1})\rightarrow(1)$$

Where $a_0,a_1,a_2....$ a^n are constants

Since the polynomial passes through $(x_0,f(x_0))$, $(x_1,f(x_1))......(x_n,f(x_n))$. The constants can be determined by substituting one of the values of $x_0,x_1,.....x_n$ *for* x in the above equation

Putting $x=x_0$ in (1) we get, $f(x_0)=a_0(x-x_1)(x_0-x_2)(x_0-x_n)$

$$\Rightarrow a_0=\frac{f(x_0)}{(x-x_1)(x_0-x_2)....(x_0-x_n)}$$

Putting $x=x_1$ in (1) we get, $f(x_1)=a_1(x-x_0)(x_1-x_2)----(x_1-x_n)$

$$\Rightarrow a_1=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

Similarly substituting $x=x_2$ in (1), we get

$$\Rightarrow a_2=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)}$$

Continuing in this manner and putting *x* = x_n in (1) we get

$$a_n=\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)----(x_n-x_{n-1})}$$

Substituting the values of $a_0,a_1,a_2....a_n$, we get

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).........(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$
$$f(x_1)+\frac{(x-x_0)(x-x_1)(x-x_2).....(x-x_n)}{(x_2-x_0)(x_2-x_1)......(x_2-x_n)}+.....f(x_2)+\frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_1)(x_n-x_2).....(x_n-x_{n-1})}f(x_n)$$

UNIT-III
Numerical Differentiation and Integration

Objectives:

- To introduce various numerical methods to find approximate derivatives.
- To introduce various numerical methods to evaluate definite integrals.

Syllabus:

Approximation of derivative using Newton's forward and backward formulas. Integration using Trapezoidal and Simpson's rules.

Learning Outcomes:

After completion of the unit, Students will able to

- determine approximate derivatives by using appropriate numerical methods
- Evaluate definite integrals using appropriate Trapezoidal and Simpson's rules.

Introduction:

Suppose a function y = f(x) is given by a table of values (x_i, y_i). The process of computing the derivative $\frac{dy}{dx}$ for some particular value of x is called Numerical differentiation.

Derivatives using Newton's forward difference formula

Newton's interpolation formula for equal intervals is

Suppose that we are given a set of values (x_i, y_i), i = 0,1,2,...., n.

We want to find the derivative of y = f (x) passing through the (n + 1)points, at a point nearer to the starting value at x = x_0.

Newton's Forward Difference Interpolation Formula is

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots \quad \text{------------------ (1)}$$

$$\text{Where } p = \frac{x - x_0}{h} \quad \text{------------------ (2)}$$

On differentiation (1) w.r.t., p we have

On differentiation (2) w.r.t. x we have, $\frac{dp}{dx} \approx \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{\mathrm{h}} \left[\begin{array}{l} \Delta y 0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6} \Delta^3 y_0 \\ + \frac{4p^3 - 18p^2 + 22p - 6}{24} \Delta^4 y_0 + \ldots \end{array} \right] \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) gives the value of $\frac{dy}{dx}$ at any point x which may be anywhere in the interval.

At x = x_0 and p = 0, hence putting p = 0, equation (3) gives

$$\left(\frac{dy}{dx}\right)_{x \approx x_1} = \left(\frac{dy}{dp}\right)_{p \approx 1} = \frac{1}{\mathrm{h}} \left[\begin{array}{l} \Delta y 0 + \frac{1}{2} \Delta^2 y_0 + \frac{1}{6} \Delta^3 y_0 \\ + \frac{4p^3 - 18p^2 + 22p - 6}{24} \Delta^4 y_0 + \ldots \end{array} \right] \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Again on differentiation (3) we get

$$\frac{d^2 y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(p-1)}{} \Delta^3 y_0 + \frac{6p^2 - 18p + 11}{12} \Delta^4 y_0 + \ldots\right]$$

From which we obtain

$$\left(\frac{d^2 y}{dx^2}\right) x \approx x_0 = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + ..\right] \text{ at x = } x_0 \text{ and p = 0 } \ldots\ldots \text{ (5)}$$

$$\text{Similarly, } \left(\frac{d^3 y}{dx^3}\right) x \approx x_0 = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \ldots\ldots\right] \qquad \ldots\ldots \text{ (6)}$$

Derivatives using Newton's Backward Difference Formula:

Newton's Backward Difference Interpolation Formula is

$$\mathrm{y(x)} = \mathrm{y_n} + \mathrm{p}\,\Delta \mathrm{y_n} + \frac{p(p+1)}{2!} \Delta^2 y_n + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_n + \ldots\ldots\ldots\ldots \quad \text{------------------ (7)}$$

$$\text{Where p} = \frac{x - x_n}{h} \qquad \text{------------------ (8)}$$

On differentiation (7) w.r.t., p we have

$$\frac{dy}{dp} = \left| \Delta y_n + \frac{2p+1}{2}\Delta^2 y_n + \frac{3p^2+6p+2}{6}\Delta^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Delta^4 y_n + \ldots \right| .$$

On differentiation (8) w.r.t. x we have, $\frac{dp}{dx} \approx \frac{1}{h}$ Now

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\Delta^2 y_n + \frac{3p^2+6p+2}{6}\Delta^3 y_n + \frac{4p^3+18p^2+22p+6}{24}\Delta^4 y_n + \ldots \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Equation (9) gives the value of $\frac{dy}{dx}$ at any point x which may be anywhere in the interval.

At $x = x_n$ and p = 0, hence putting p = 0, equation (9) gives

$$\left(\frac{dy}{dx}\right)x \approx x_{n_1} = \left(\frac{dy}{dx}\right)x_n = \frac{1}{h}\left[\Delta yn + \frac{1}{2}\Delta^2 y_n + \frac{1}{3}\Delta^3 y_n + \frac{1}{4}\Delta^4 y_n + \ldots \right] \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Again on differentiation (09) we obtain

$$\frac{d^2 y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}\cdot\frac{dp}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_n + \frac{(p+1)}{}\Delta^3 y_n + \frac{6p^2+18p+11}{12}\Delta^4 y_n + . \right]$$

From which we obtain

$$\left(\frac{d^2 y}{dx^2}\right) x \approx x_n = \frac{1}{h^2}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + .. \right]$$ at $x = x_n$ and p = 0

Similarly, $\left(\frac{d^3 y}{dx^3}\right) x \approx x_n = \frac{1}{h^3}\left[\Delta^3 y_n - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots \right]$ .... (12)

Problem 1. Find $\frac{dy}{dx}$ *and* $\frac{d^2 y}{dx^2}$ *at* $x = 51$ from the following data.

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Solution: Here h = 10. To find the derivatives of y at x = 51 we use Newton's Forward difference formula taking the origin at a = 50.

We have $p=\dfrac{x-x_0}{h}=\dfrac{51-50}{10}0.1$

$$\left(\frac{dy}{dx}\right)_{x=51}=\left(\frac{dy}{dx}\right)_{p=0.1}=\frac{1}{h}\left[\Delta y_0+\frac{(2p-1)}{2!}\Delta^2 y_0+\frac{(3p^2-6p+2)}{3!}\Delta^3 y_0+\frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0+\ldots\right]$$

The difference table is given by

x	$p=\dfrac{x-50}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	**19.96**				
			16.69			
60	1	36.65		**5.47**		
			22.16		**-9.23**	
70	2	58.81		-3.76		**11.99**
			18.40		2.76	
80	3	77.21		-1.00		
			17.40			
90	4	94.61				

$$\therefore\left(\frac{dy}{dx}\right)_{p=0.1}=\frac{1}{10}\left[16.69+\frac{(0.2-1)}{2}(5.47)+\left[\frac{3(0.1)^2-6(0.1)+2}{6}\right](-9.23)+\frac{\left[4(0.1)^3-18(0.1)^2+22(0.1)-6\right]}{24}\times 11.99+\ldots\right]$$

$$=\frac{1}{10}\left[16.69-2.188-2.1998-1.9863\right]=1.0316$$

$$\left(\frac{d^2y}{dx^2}\right)_{p=0.1}=\frac{1}{h^2}\left[\Delta^2 y_0+(p-1)\Delta^3 y_0+\frac{(6p^2-18p+11)}{12}\Delta^4 y_0+\ldots\right]$$

$$=\frac{1}{100}\left[5.47+(0.1)-1(-9-23)+\frac{\left[6(.1)^2-18(.1)+11\right]}{12}\right]\times 11.99$$

$$=\frac{1}{100}\left[5.47+8.307+9.2523\right]$$

=0.2303.

Problem 2. The population of a certain town is shown in the following table

Year x	1931	1941	1951	1961	1971
Population y	40.62	60.80	79.95	103.56	132.65

Find the rate of growth of the population in 1961.

Solution. Here h = 10 Since the rate of growth of population is $\dfrac{dy}{dx}$ we have to find $\dfrac{dy}{dx}$ at x = 1961, which lies nearer to the end value of the table. Hence we choose the origin at x = 1971 and we use Newton's backward interpolation formula for derivative.

$$\frac{dy}{dx}=\frac{1}{h}\left[\nabla y_4+\frac{(2p+1)}{2}\nabla^2 y_4+\frac{(3p^2+6p+2)}{6}\nabla^3 y_4+\frac{(2p^3+9p^2+11p+3)}{12}\nabla^4 y_4+\ldots\right]$$

Where $p=\frac{x-x_0}{10}=\frac{1961-1971}{10}=-1$

The backward difference table

x Year	y Population	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	**-4.47**
1951	79.95		4.46		
		23.61		**1.02**	
1961	103.56		**5.48**		
		29.09			
1971	132.65				

$$\left(\frac{dy}{dx}\right)_{p=-1}=\frac{1}{10}\left[29.09+-\left(\frac{1}{2}\right)(5.48)+\frac{\left[3(-1)^2+6(-1)+2\right]}{6}\times 1.02+\frac{\left[2(-1)^3+9(-1)^2+11(-1)+3\right]}{12}(-4.47)\right]$$

$$=\frac{1}{10}\left[29.09-2.74-0.17+0.3725\right]$$

$$=\frac{1}{10}\left[26.5525\right]=2.6553$$

$\therefore$ The rate of growth of the population in the year 1961 is 2.6553.

Maxima and Minima of a tabulated Function:

Given a set of data points (x_i, y_i) , i = 0,1,2,....., n, we can get the interpolating polynomial of degree n. Now we wish to estimate the value of x at which the curve is maximum or minimum.

We know that the maximum and minimum values of a function can be determined by equating the first derivative to zero and solving for the variable. The same procedure can be applied to find the maxima and minima of a tabulated function. Assume that the points are equally spaced with a step size of h.

Consider Newton's forward difference interpolation formula

$y \approx y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-3)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$ On differentiation it w.r.t. p, we get $\frac{dy}{dp} = \left[\Delta y_n + \frac{2p-1}{2}\Delta^2 y_n + \frac{3p^2-6p+2}{6}\Delta^3 y_n + ..\right]$ ....(1)

For maxima and minima $\frac{dy}{dp} \approx 0$. hence equating the RHS of (1) to zero and for simplicity only upto 3rd differences we obtain

$$\left[\Delta y_n + \frac{2p-1}{2}\Delta^2 y_n + \frac{3p^2-6p+2}{6}\Delta^3 y_n + ..\right] = 0$$

Re-arranging this as a quadratic in p, we get

$$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + \left(\Delta^2 y_0 - \Delta^3 y_0\right)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \frac{1}{3}\Delta^3 y_0$ from the difference table, We solve the equation (2) for p. Then the corresponding values of x are given by $x = x_0 + ph$ at which y is maximum or minimum.

Numerical Integration

Given set of (n + 1) data points (x_i, y_i), i = 0,1,2,.......,n of the function y = f(x), where f(x) is not known explicitly, it is required to evaluate $\int_{x_0}^{x_n} f(x)\,dx$.

Newton-Cote's Quadrature Formula (General Quadrature Formula):

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods known as Newton-Cote's methods.

Derivation of Newton-Cotes formula:

Let the interval [a, b] be divided into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < x_3 \ldots\ldots\ldots < x_n = b$. Then $x_n = x_0 + nh$.

Newton forward difference formula is

$$y(x) = y(x_0 + ph) = P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Where p = $\dfrac{x \quad x_0}{h}$ .Now, instead of f(x) we will replace it by this interpolating polynomial.

$\therefore \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx$, where $P_n(x)$ is an interpolating polynomial of degree n

$$= \int_{x_0}^{x_0+nh} P_n(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \right] dx$$

Since $x = x_0 + ph$, $dx = h.dp$ and hence the above integral becomes

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \right] dp$$

$$= h\left[y_0(p) + \frac{p^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - 3\frac{p^3}{3} + 2\frac{p^2}{2}\right)\Delta^3 y_0 + ... \right]$$

$$= h\left[ny_0 + \frac{n^2\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - 3\frac{n^3}{3} + 2\frac{n^2}{2}\right)\Delta^3 y_0 + ... \right]$$

$$= nh\left[y_0 + \frac{n\Delta y_0}{2} + \frac{1}{2}\left(\frac{n^2}{3} - \frac{n}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^3}{4} - 3\frac{n^2}{3} + 2\frac{n}{2}\right)\Delta^3 y_0 + ... \right]$$

This is called **Newton-Cote's Quadrature for)mula.** ....(2)

Trapezoidal Rule:

Putting n = 1 in the above general formula, all differences higher than the first will become zero (since other differences do not exist if n = 1) and we get

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right] = \frac{h}{2}(y_0 + y_1)$$

and $\displaystyle\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1 \right] = h\left[y_1 + \frac{1}{2}(y_2 - y_1) \right] = \frac{h}{2}(y_1 + y_2)$

$$\int_{x_2}^{x_3} f(x)\,dx = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = h\left[y_2 + \frac{1}{2}\Delta y_2 \right] = h\left[y_2 + \frac{1}{2}(y_3 - y_2) \right] = \frac{h}{2}(y_2 + y_3)$$

....

Finally,

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Hence,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+3h} f(x)\,dx + \ldots.. + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \ldots\ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}[(y_0 + y_1) - 2(y_1 + y_2 + y_3 + y_4 + \ldots\ldots + y_{n-2} + y_{n-1}]$$

.... (3)

Simpson's 1/3 Rule

Putting n = 2 in Newton-Cotes Quadrature formula i.e., by replacing the curve y = f(x) by n/2 parabolas, we have

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0\right] = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right] \quad = 2h\left[\frac{1}{6}y_0 + \frac{2}{3}y_1 + \frac{1}{6}y_2\right]$$

$$= \frac{2h}{6}[y_0 + 4y_1 + y_2] = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$

....

$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$ Adding all these integrals, we obtain

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \ldots\ldots\ldots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \ldots\ldots\ldots + \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots\ldots\ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + y_3 + y_5 + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots\ldots + y_{n-2})]$$

.... (4)

$$= \frac{h}{3}\left[\begin{array}{l} \text{sum of the first and last ordnates}) + 4(\text{sum of the odd ordinates}) \\ + 2(\text{sum of the remaining even ordinates} \end{array}\right]$$

With the convention that $y_0, y_2, y_4, \ldots.., y_{2n}$ are even ordinates and $y_1, y_3, y_5, \ldots.., y_{2n-1}$ are odd ordinates.

This is known as **Simpson's 1/3 rule** or simply **Simpson's rule**.

Simpson's 3/8 Rule:

n = 3 in Newton-Cote's Quadrature formula, all differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6\quad 3)}{12}\Delta^2 y_0 + \frac{3(3\quad 2)^2}{24}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = 3\text{h}\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}\text{h}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$\int_{x_3}^{x_6} f(x)\,dx = \frac{3}{8}\text{h}[y_3 + 3y_4 + 3y_5 + y_6]$ and so on.

Adding all these integrals, from x_0 to x_n, where n is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + ... + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + ... + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$= \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1}) + 2(y_3 + y_6 + y_9 + ... + y_n)]$$

.... (5)

Equation (5) is called **Simpson's 3/8 rule** which is applicable only when n is multiple of 3.

Problems : Evaluate $\int_0^1 \frac{dx}{1+x}$ using (i) Trapezoidal rule (ii) Simpson's one third rule (iii) Simpson's three eight rule. Take $h = \frac{1}{6}$ for all cases.

Solutions: Here $h = \frac{1}{6}$, Let $y = f(x) = \frac{1}{1+x}$. The values of f(x) for the points of subdivisions are as follows:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1	0.8571	0.75	0.6667	0.6	0.5455	0.5

(i) Tapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$\square\ \frac{1}{12}\left[\left(1+0..5\right) + 2\left(0.8571 + 0.755 + 0.6667 + 0.6 + 0.5455\right)\right]$$

=0.6949.

(ii) Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[\left(y_0 + y_6\right) + 2\left(y_2 + y_4\right) + 4(y_1 + y_3 + y_5)\right]$$

$$\square\ \frac{1}{18}\left[\left(1+0.5\right) + 2\left(0.75 + 0.6\right) + 4(0.8571 + 0.6667 + 0.5455)\right]$$

= 0.6932.

(iii) Simpson's three eight rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2y_3\right]$$

=1 / 16[(1+0.5)+3(0.8571+0.75+0.6+0.5455)+2(0.6667)]

= 0.6932.

Assignment-Cum-Tutorial Questions

SECTION-A

Questions testing the remembering / understanding level of students

I) Objective Questions

1. If we put n = 2 in a general quadrature formula, we get
   (a) Trapezoidal rule (b) Simpson's 1/3rd rule
   (c) Simpson's 3/8th rule (d) Boole's rule
2. In Simpson's 1/3rd rule the number of subintervals should be
   (a) Even (b) odd (c) multiples of 3's (d) more than 'n' interval
3. If *f(2) = 5, f(4) = 8, f(6) = 10, and f(8) = 16* then *f''(8) = ?*
4. If the distance *d(t)* is traversed by a particle in the 't' sec and *d(0) = 0, d(2) = 8, d(4) = 20 and d(6) = 28*, then its velocity in cm after 6 secs is
   (a) 1.67 (b) 16.67 (c) 2 (d) 2.003
5. The formula $\frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + ...\right]$ is used only when the point x is at
   (a) end of the tabulated set (b) middle of the tabulated set
   (c) Beginning of tabulated set (d) none of these
6. Numerical differentiation gives
   (a) exact value (b) approximate value (c) no result (d) negative value
7. The general quadrature formula is
   (a) depends upon interpolation formula (b) always same
   (c) not easy to derive (d) is also given approximate result
8. For n =1in quadrature formula, $\int_{x_0}^{x_1} f(x)\,dx$ equals to
   (a) $\frac{h}{2}(f_0 + f_1)$ (b) $(f_0 + f_1)$ (c) $\frac{h}{2}(f_0 - f_1)$ (d) $\frac{h}{4}(f_0 + f_1)$
9. To apply, Simpson's 1/3rd rule, always divide the given range of integration into 'n' parts ,where n is
   (a) even (b) odd (c) 1,2,3,4 (d) 1,3,5,7

10. The process of calculating derivative of a function at some particular value of the independent variable by means of a set of given values of that function is

    (a) Numerical value (b) Numerical differentiation

    (c) Numerical integration (d) quadrature

11. In the second derivative using Newton's backward difference formula, the coefficient of $\nabla^2 f(a)$

    (a) $-1/h^2$ (b) $1/h^2$ (c) 11/12 (d) $-h^2$

12. While evaluating definite integral by Trapezoidal rule, the accuracy can be increased by

    (a) large number of sub-intervals (b) even number of sub-intervals

    (c) h =4 (d) multiples of 3

13. Numerical integration when applied to a single variable function, is called as

    (a) maxima (b) minima (c) quadrature (d) quadrant

14. To increase the accuracy in evaluating a definite integral by Trapezoidal rule, we should take ____________________

15. Values of y = f(x) are known as $x = x_0, x_1$ and x_2. Using Newton's forward integration formula, the approximate value of $\left(\frac{dy}{dx}\right)_{x=x_0}$ is _________

SECTION-B

Descriptive Questions:

1. A chemical company, wishing to study the effect of extraction time on the efficiency of an extraction operation, obtained the data shown in the following table.

Extraction time minutes(x)	27	45	41	19	3	39	19	49	15	31
Efficiency(y)	57	64	80	46	62	72	52	77	57	68

   Fit a straight line to the given data by the method of least squares.

2. The velocity of a train which starts from rest is given by the following table being reckoned in minutes from the start and speed in miles per hour

Minutes	2	4	6	8	10	12	14	16	18
Miles per hour	10	18	25	29	32	20	11	5	2

Estimate approximately the total distance travelled in 20 minutes.

3. Evaluate $\int_0^2 e^{-x^3} dx$ using Simpson's rule taking h=0.25

4. The following table gives the velocity v of a particle at time 't'

t (seconds)	0	2	4	6	8	10	12
v meters per second	4	6	16	34	60	94	136

Find (i)the distance moved by the particle in 12 seconds and also (ii) the acceleration at t = 2 sec

5. A river is 80 meters wide. The depth 'd' in meters at a distance x from the bank is given in the following table. Calculate the cross section of the river using Trapizoidal rule.

x	10	20	30	40	50	60	70	80
d(x)	4	7	9	12	15	14	8	3

6. A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and the position of the rocket at t=40 seconds use Trapezoidal rule

t	0	5	10	15	20	25	30	35	40
d(t)	40	40.25	48.50	51.25	54.35	59.48	61.50	64.30	68.70

7. A curve is expressed by the following values of x and y. Find the slope at the point x =1.5

x	0.0	0.5	1.0	1.5	2.0
y	0.4	0.35	0.24	0.13	0.05

Question testing the ability of students in applying the concepts.

1. A rod is rotating in a plane. The following table gives the angle θ(in radians) through which the rod has turned for various values of the time 't'

t	0.0	0.2	0.4	0.6	0.8	1.0
θ	0.00	0.12	0.49	1.12	2.02	3.20

2. In a machine a slider moves along a fixed straight rod. Its distance x cms along the rod is given below for various values of time't' seconds. Find the velocity and acceleration of the slider when t = 0.3

t(sec)	0	0.1	0.2	0.3	0.4	0.5	0.6
x(cms)	30.13	31.62	32.87	33.64	33.95	33.81	33.24

3. The population of a certain town is given below. Find the rate of growth of the population in 1961:

Year	1931	1941	1951	1961	1971
Population	40.62	60.80	71.95	103.56	132.65

4. The distance covered by an athlete for the 50 meter is given in the following table

Time(sec)	0	1	2	3	4	5	6
Distance(meter)	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of the athlete at t = 5 sec. correct to two decimals.

5. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3rd rule, find the velocity of the rocket at t= 80 seconds.

t sec	0	10	20	30	40	50	60	70	80
f(cm/sec^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

6. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h(ft)	10	11	12	13	14
A(sq.ft)	950	1070	1200	1350	1530

7. A curve is drawn to pass through the points given by following table:

x	1	1.5	2.0	2.5	3	3.5	4.0
y	2	2.4	2.7	2.8	3	2.6	2.1

Questions testing the analyzing / evaluating ability of students

1. Using Simpson's 1/3rd rule, find the value of the integral $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ by taking 6 sub-intervals.
2. Minimum number of subintervals required to evaluate the integral $\int_{1}^{2} \frac{1}{x} dx$ by using Simpson's 1/3rd rule so that the value is corrected up to 4 decimal places.

LEARNING RESOURCES:

1. B.S. Grewal, Numerical Methods in engineering and science, Khanna Publishers, 43rd edition, 2014.
2. Dr. M.K. Venkataraman, Numerical Methods in Science and Engineering, National Publishing Co., 2005.
3. V. Ravindranadh, P. Vijayalakshmi, A text book on Mathematical methods, Himalaya publishing house, 2011.

REFERENCE BOOKS/OTHER READING MATERIAL:

1. S.S. Sastry, Introductory Methods of Numerical Analysis, 4th edition, 2005.
2. E. Balagurusamy, Computer Oriented Statistical and Numerical Methods – Tata McGraw Hill., 2000.
3. M.K.Jain, SRK Iyengar and R.L.Jain, Numerical Methods for Scientific and Engineering Computation, Wiley Eastern Ltd., 4th edition, 2003.
4. S. Arumugam, A. Thangapandi Isaac, A. Somasundaram, Numerical methods , 2nd Edition, Scitech publications, 2005.
5. T.K.V. Iyenger, B. Krishna garndhi, S. Ranganadham, M.V.S.S.N. Prasad, Mathematical methods, Revised Edition, 2012.

-----------------@-@-@-@-@---------------

UNIT-5

Complex differentiation & Integration (without proofs)

Objectives:

- To introduce the basic theory of functions of a complex variables, analytic functions, Harmonic functions and the C-R equations which play a vital role in several engineering Problems.
- Exposed the methods to evaluate complex integrals.
- Student learns the methods to evaluate integrals by applying Cauchy's integral formula and its generalization

Syllabus:

- Introduction of functions of a complex variables
- Properties of analytic function
- Cauchy-Riemann equations in Cartesian form-problems
- Cauchy-Riemann equations in polar form-problems
- Harmonic and conjugate harmonic functions-problems
- Introduction of line integral
- evaluation along the path
- Cauchy's integral theorem
- Cauchy's integral formula and Generalized integral formula

Outcomes: Student will be able to:

1. Test the analyticity of complex functions by applying C-R Equations
2. Transform ordinary function into Analytical function using procedures of conjugate harmonic function.
3. Evaluate Line integral of the complex functions
4. Apply Cauchy's integral formula to solve integration problems in engineering

Introduction:

Complex analysis traditionally known as the **theory of functions of a complex variable** is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatory, applied mathematics; as well as in physics, including hydrodynamics and thermodynamics and also in engineering fields such as nuclear, aerospace, mechanical and electrical engineering. We have used complex numbers in a number of situations, and in general, complex analysis is used in many areas of electrical engineering including Circuit theory (impedance, transfer functions, *etc*.),Electromagnetism (time-harmonic fields),Electrostatics (solutions to Laplace's equation), and Electromagnetic.

Complex analysis is particularly concerned with analytic functions of complex variables (or, more generally, meromorphic functions). Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

Functions of complex variable: Suppose ‘S’ is a set of complex numbers. A function defined and which assigns a complex number to every ‘z’ in ‘S’ is called function ‘f’ on ‘S’. And we write $f(z) = w$

Limit of a function: A function $f(z) = w$ is said to tend to limit ‘l’ as approaches a point z_0, if for every real $\epsilon > 0$ $\exists$ a positive δ such that $|f(z) - l| < \epsilon$ for $0 < |z - z_0| < \delta$.

We write $\lim_{z\to z_0} f(z) = l$

Continuity of function: A function $f(z) = w$ is said to be continuous at a of z_0 if $f(z_0)$ exists, $\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$.

Differentiability: A function $f(z) = w$ is said to be differentiable at a point z_0 if the limit

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f^{'}(z_0)$$

Limit is known as the derivative of $f(z)$ at z_0 and denoted by $f^{'}(z_0)$

(Or)

Let f be a function whose domain of definition contains a neighbourhood of a pointz_0. The derivative off at z_0, written $f'(z_0)$, is defined by the equation

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f^{'}(z_0)$$

Analytic function: A function f of the Complex variable z is analytic in an open set if it has a derivative at each point in that Set. If we should speak of a function f that is analytic in a set S which is not open, It is to be understood that f is analytic in an open set containing S. In particular, f is Analytic at a point z_0 if it is analytic throughout some neighbourhood of z_0.

For example, the function $f(z) = I/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z^2|$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighbourhood.

Note: An analytic function is also known as regular function, holomorphic function, and monogenic function.

Entire function: An entire function is a function that is analytic at each point in the entire finite Plane Since the derivative of a polynomial exists everywhere, it follows that **every Polynomial is an entire function.**

Singular function: If a function *f* fails to be analytic at a point z_0 but is analytic at some point In every neighbourhood of z_0, then z_0 is called a singular point, or singularity, of f.

For example: 1.The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$.

2. The function $f(z) = |z^2|$, on the other hand, has no singular points since it is nowhere analytic.

Cauchy- Riemann (C-R) Equations: Cauchy- Riemann (C-R) Equations are used to determine whether a complex function is analytic or not.

Cauchy- Riemann (C-R) Equations (theorem in Cartesian coordinates): Necessary and sufficient conditions for the function $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in the region R are (i) u_x, u_y, v_x, v_y are continuous in the region R (sufficient)

(ii) $u_x = v_y, u_y = -v_x$ (Necessary) the equations (ii) are known as Cauchy- Riemann (C-R) Equations.

Note: 1. C-R conditions are necessary but not sufficient.

2. C-R conditions are sufficient if the partial derivatives are continuous i.e., if $u(x,y), v(x,y)$ have continuous first partial derivatives and satisfy C-R conditions then f is analytic.

Properties of analytic functions:

1. If $f(z)$ and $g(z)$ are analytic , then $f \pm g, fg, \frac{f}{g}$ are analytic if $g(z) \neq 0$
2. Analytic function of an analytic function is analytic.
3. An entire function of an entire function is entire.
4. Derivative of an analytic function is itself analytic function
5. If $f = u + iv$ is analytic, then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are mutually orthogonal i.e., $u(x,y) = c_1$ are orthogonal trajectories of $v(x,y) = c_2$ and vice versa.
6. The real and imaginary part of an analytic function satisfy Cauchy- Riemann (C-R) Equations

Cauchy- Riemann (C-R) Equations (in Polar coordinates): If $w = f(z) = u(r,\theta) + iv(r,\theta)$ and $f(z)$ is derivable at $z = re^{i\theta}$ then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$ or $ru_r = v_\theta, rv_r = -u_\theta$.

Proof: Let $z = x + iy = re^{i\theta} = r(x\cos\theta + i\sin\theta)$

$$\Rightarrow x = \cos\theta, y = \sin\theta$$

$$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \qquad \text{------------- (1)}$$

Differentiating w. r. to r, we get

$$f'(r.e^{i\theta}).e^{i\theta} = u_r + iv_r \qquad \text{------------- (2)}$$

Differentiating w. r. to θ, we get

$$f'(r.e^{i\theta}).rie^{i\theta} = u_\theta + iv_\theta \qquad \text{-------------- (3)}$$

Substituting equation (2) in (3), we get

$$ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}.$$

Hence these are the Cauchy- Riemann (C-R) Equations in Polar coordinate.

Problems:

1. Show that $f(z) = xy + iy$ is everywhere continuous but it is not analytic.

Sol: Given that $f(z) = xy + iy$

We have $z = x + iy \Rightarrow z_0 = x_0y_0 + iy_0$

Now $\lim_{z\to z_0} f(z) = \lim_{z\to z_0}(xy+iy)$

$$= \lim_{\substack{x\to x_0 \\ y\to y_0}}(xy+iy)$$

$$= x_0y_0 + y_0$$

$$= f(z_0)$$

$$\therefore \lim_{z\to z_0} f(z) = f(z_0)$$

$\therefore f(z)$ is continuous everywhere .

$f(z) = xy + iy = u + iv$

Here $u = xy$ and $v = y$

$u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 1$

$\therefore u_x \neq v_y$ and $v_x \neq -u_y$

$\therefore f(z)$ Is not analytic

2. Prove that $f(z) = \bar{z}$ is not analytic at any point.

Sol: Given that $f(z) = \bar{z}$

We have $z = x + iy \Rightarrow \bar{z} = x - iy$

Here $u = x$ and $v = -y$

$u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$

$\therefore u_x \neq v_y$ and $v_x = -u_y$

$\therefore f(z)$ Is not analytic

3. Find whether $f(z) = \dfrac{x-iy}{x^2+y^2}$ is analytic or not.

Sol: Given that $f(z) = \dfrac{x-iy}{x^2+y^2} = u + iv$

Here $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).1 - x(2x+0)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2).0 - (-y)(2x+0)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).0 - x(2y+0)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2).1 - (-y)(2y+0)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore u_x = v_y$ and $v_x = -u_y$

Hence $f(z)$ is analytic.

Exercise problems:

1. Prove that z^n is analytic and hence find its derivative.
2. Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.
3. Find whether $w = logz$ is analytic or not find its derivative
4. Find all values of$k$, such that $f(z) = e^x(cosky + isinky)$ is analytic.
5. Find where the function (i) $w = \dfrac{1}{z}$ $(ii)\dfrac{z}{z-1}$ fails to be analytic.

Definitions:

Laplace equation: The equation $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 or \nabla^2\phi=0$ where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is called Laplace equation. ∇^2 Is called Laplace operator

Harmonic Function: A function $f(x,y)$ is said to be a harmonic function if it satisfies the Laplace equation i.e. $\nabla^2 f=0$.

Conjugate Harmonic Function: The real part u of an analytic function $f = u + iv$ is known as the Conjugate Harmonic Function of v and vice versa i.e. v is Conjugate Harmonic Function of u(u is Conjugate Harmonic Function of v.

Problems:

1. Show that $u = e^{-x}(xsiny - ycosy)$ is harmonic.

   Sol: We have $u = e^{-x}(xsiny - ycosy)$

   Differentiating partially w.r.to x and y, we get

   $u_x = e^{-x}(siny) - e^{-x}(xsiny - ycosy)$

   $u_{xx} = -2e^{-x}(siny) + xe^{-x}siny - e^{-x}ycosy$

   And

   $u_y = e^{-x}(xcosy + ysiny - cosy)$

   $u_{yy} = 2e^{-x}(siny) - xe^{-x}siny + e^{-x}ycosy$

   Hence $u_{xx} + u_{yy} = 0$

   $\therefore u$ is harmonic

2. Find k such that $f(x,y) = x^3 + 3kxy^2$ be harmonic and find its conjugate.

   Sol: We have $f(x,y) = x^3 + 3kxy^2$

   $f_x = 3x^2 + 3ky^2$, $f_y = 6xky$

   $f_{xx} = 6x$, $f_{yy} = 6kx$

   Since $f(x,y)$ is harmonic, $\therefore f_{xx} + f_{yy} = 0$

   $\Rightarrow 6x + 6kx = 0 \Rightarrow k = -1 \quad (\because x \neq 0)$

   Hence $f(x,y) = x^3 - 3xy^2$

   Let $g(x,y)$ be the conjugate of $f(x,y)$.

   $$dg=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy$$

   $$=-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}dy$$ (Using C- R equations since $f(z)$ is analytic)

   $$=-6kxydx+(3x^2+3ky^2)dy$$

   $$\Rightarrow dg=6xydx+(3x^2-3y^2)dy$$

   This is exact differential equation. Integrating, we get

   $$g=\int_{y constant} 6xydx+\int_{without x terms} -3y^2dy+c$$

   $$=6y\left(\frac{x^2}{2}\right)-3\left(\frac{y^3}{3}\right)+c \Rightarrow 3x^2y-y^3+c$$

   $g(x,y)$ is $3x^2y-y^3+c$

Exercise problems:

1. Show that $u(x,y) = x^3 - 3xy^2$ is harmonic
2. Show that $f(x,y) = x^3y - xy^3 + xy + x + y$ can be the imaginary part of an analytic function.
3. Show that the function $u = 2\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate.

Exercise problems:

1. Find the analytic function whose real part is $x^2 - y^2 - y$

2. Find the analytic function whose imaginary part is $x^2 - y^2 + \frac{x}{x^2+y^2}$

Introduction:
Complex integration forms the cornerstone of complex variable theory. The key results in Chapter are the Cauchy–Goursat theorem and the Cauchy integral formulas. Other interesting results include (Gauss' mean value theorem, Liouville's theorem and the maximum modulus theorem). The link of analytic functions and complex integration with the study of conservative fields is considered. Complex variable methods are seen to be effective analytical tools to solve conservation field models in potential flows, gravitational potentials and electrostatics.

Definitions:

Simple curve: Simple curve is a curve having no self intersections i.e., no two distinct values of 't' correspond to the same point (x, y)

Closed curve: Closed curve is one in which end point coincide i.e.,$\emptyset(a)$ = $\emptyset(b)$ and $\varphi(a)$ = $\varphi(b)$.

Simple closed curve: Simple closed curve **is** a curve having no self intersection and with coincident end point.

Contour: Contour is a continuous chain of a finite number of smooth arcs

Closed contour is a piecewise smooth closed curve without point of self intersection.

Examples: Boundaries of circle, ellipse, rectangle, triangle.

Line Integral: Definite integral or complex line integral or simply line integral of a complex function $f(z)$

From $z_1 to z_2$ along a curve 'C' is defined as $\int_c f(z)dz$ here 'C' is known as the path of integration .If it is a closed curve, the line integral is denoted by $\oint_c$.

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy)$$

Basic Properties of Line Integrals

1. **Linearity:** $\int_c (c_1 f(z) + cl2g(z))dz = c_1 \int_c f(z)dz + c_2 \int_c g(z)dz$
2. **Sense reversal:** $\int_A^B f(z)dz = -\int_B^A f(z)dz$
3. **Partitioning of path:** $\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{C_2} f(z)dz$

   Where curve c consists of the curves c_1 $and c_2$

Evaluation of complex line integral:

1. By indefinite integration (of analytic function): If $f(z)$ is analytic in a simply connected domain 'D' then $\int_{z_2}^{Z_1} f(z)dz = F(z_2) - F(z_1)$ where $F^{'}(z) = \frac{dF}{dz} = f(z)$ in D.
2. By use of the path: If curve 'C' is represented by $z = z(t), a \le t \le b$ then $\int_c f(z)dz = \int_a^b f(z(t))\frac{dz}{dt}dt$ i.e. , the integral is converted to an ordinary integral in t by making use of the property of the curve c.

Problems on line integral:

1. Evaluate $\int_0^{1+i}(x^2+iy)dz$ along the path (i) $y = x$ (ii) $y = x^2$

Sol: We known that $z = x + iy \Rightarrow dz = dx + idy$

(i) Along the line = ⇒ = and x varies from 0 to 1.

$$\int_0^{1+i}(x^2+iy)dz = \int_{(0,0)}^{(1,1)}(x^2+iy)(dx+idy)$$

$$= \int_0^1 (x^2+ix)(dx+idx) \quad \text{(since =)}$$

$$= (1+i)\int_0^1 (x^2+ix)dx \Rightarrow (1+i)\left(\left(\frac{x^3}{3}+i\frac{x^2}{2}\right)\right)_0^1$$

$$= (1+i)(\frac{1}{3}+\frac{i}{2})$$

(ii) Along the parabola = ^{2} ⇒ = *2* and x varies from 0 to 1

$$\int_0^{1+i}(x^2+iy)dz = \int_{(0,0)}^{(1,1)}(x^2+iy)(dx+idy)$$

$$= \int_0^1 (x^2+ix^2)(dx+i2xdx) \quad \text{(since = }^2\text{)}$$

$$= \int_0^1 (x^2(1+i)(1+2xi)dx \Rightarrow (1+i)\int_0^1 x^2(1+2xi)dx$$

$$= (1+i)\int_0^1 (x^2+2x^3 i)dx$$

$$= (1+i)\left(\left(\frac{x^3}{3}+2i\frac{x^4}{4}\right)\right)_0^1$$

$$= (1+i)(\frac{1}{3}+\frac{i}{2})$$

2. Evaluate $\int (2y+x^2)dx+(3x-y)dy$ along the parabola = *2*, = *²* + *3* joining (*0*,*3*) (*2*,*4*).

   Sol: Given that = *0*, = *3*, = *0* = *2* , = *4*, = *1*

   Substituting for x and y in terms of t, i.e., = *2*, = *²* + *3* ⇒ = *2*, = *2* we get

$$I = \int_{t=0}^{1} [2(t^2+3)+4t^2]2dt + \int_{t=0}^{1} [6t-t^2-3]2tdt$$

$$= \int_{0}^{1} (24t^2-2t^3-6t+12)dt$$

$$= \left(\left(\frac{24t^3}{3}-\frac{2t^4}{4}-\frac{6t^2}{2}+12t\right)\right)_0^1$$

$$= \frac{33}{2}$$

Exercise problems:

1. Evaluate $\oint_c |z^2| dz$ around the square with vertices at (*0*,*0*), (*1*,*0*), (*1*,*1*) (*0*,*1*).
2. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)dz$ along the straight line joining (*1*, −) (*2*,).
3. Evaluate $\int_{0}^{1+i} (x-y+ix^2)dz$ (i) along the straight line from = *0* = *1* +

   (ii) Along the real axis from = *0* = *1* and then along a line parallel to imaginary axis from = *0* = *1* +

4. Evaluate $\int_{0}^{3+i} z^2 dz$ along parabola = *3²*

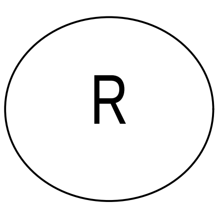
Definitions:

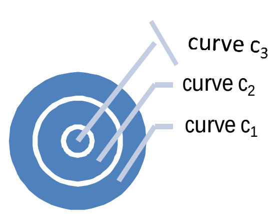
Simple curve: A curve which does not intersect itself. A simple curve which is closed is called **simple closed curve.**

Multiple curves: A curve which crosses itself.

Connected region: A region is said to be **connected region** if any two points of the region can be connected by a curve which lies entirely within the region.

Multiply connected region: A region which is bounded by more than one curve is called a **multiply connected region.**





R-- Simple connected region, multiply connected region by more than one curve

Cauchy- Goursat Theorem: A function $f(z)$ is analytic at each point inside and on a simple closed curve 'c', then $\int_c f(z)dz = 0$.

Cauchy's (Integral) Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be analytic on and within a simple closed curve contour 'c' and let $f^{'}(z)$ be continuous there. Then, $\int_c f(z)dz = 0.$

Problems on Cauchy integral theorem:

1. Evaluate $\int_c \frac{e^{2z}}{z-2}dz$ where C is $|z| = 1$

   Sol: Given that $|z| = 1$ means the circle having radius one with centre (0, 0)

   $\therefore$ The point $z = 2$ lies outside C.

   But by Cauchy integral theorem it is analytic within and on C

   Hence by Cauchy's theorem, $\int_c \frac{e^{2z}}{z-2}dz = 0$.

2. Verify Cauchy's theorem for the function $f(z) = z^2 + 3z - i2$ if C is the circle $|z| = 1$

   Sol: Let C: $z = re^{i\theta}$ where $0 \le \theta \le 2\pi$.

   $$z = e^{i\theta}, dz = ie^{i\theta}d\theta \because r = 1$$

   $$\therefore \quad \int_c f(z)dz = \int_0^{2\pi} (e^{2i\theta} + 3e^{i\theta} - 2i)(ie^{i\theta})d\theta$$

   $$= i\int_0^{2\pi} (e^{3i\theta} + 3e^{2i\theta} - 2ie^{i\theta})d\theta$$

   $$= 0 \qquad (\because \int_0^{2\pi} e^{in\theta} = 0 if n \ne 0)$$

Exercise problems:

1. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ if C is the square with vertices at *1* ± *−1* ± .
2. Verify Cauchy's theorem for the integral of *3* taken over the boundary of the rectangle with vertices *−1*, *1*, *1* + , *−1* + .

Cauchy's integral formula: Let $f(z)$ be an analytic function everywhere on and within a closed contour 'C'. If $z = a$ is any point within 'C', then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

Where the integral is taken in the positive sense around 'C'.

Cauchy's theorem for doubly connected regions: If $f(z)$ is analytic in the doubly connected

Region bounded by the curve $c_1 and c_2$ then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Generalized Cauchy's integral formula: If $f(z)$ is on analytic on and within a simple closed curve 'C' and if 'a' is any point within 'C', then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$$

Problems on Cauchy's integral formula (including generalization):

1. Let 'C' be the circle |z|=3 described in positive sense. Let

   $f(a) = \int_c \frac{2z^2 - z - 2}{z-a}dz$ (|a|≠3)

   Show that $f(2) = 8\pi i$. what the value is of $f(a) if$ || > *3*.

   Sol: Given that || = *3* is the circle with centre at (0,0) and radius equal to 3 units.

   Consider $f(a)$ is analytic everywhere except at =

   This point maybe (i) within the circle or (ii) on the circle or(iii) outside the circle

   Since |a|≠3, = is not on the circle.

   Case I: If = is within the circle, $f(a)$ is analytic everywhere except at =

   Take $f(z) = 2z^2 - z - 2$

   $$f(a) = \int_c \frac{2z^2 - z - 2}{z-a}dz$$

   $$= \int_c \frac{f(z)}{z-a}dz$$

   $$= 2\pi i f(a)$$ (By Cauchy's integral formula)

   $$= 2\pi i(2a^2 - a - 2)$$

   $$\therefore f(2) = 2\pi i(8-2-2) = 8\pi i.$$

   Case II: If|| > *3*, = is outside the circle|| = *3*.

   $\therefore f(a)$ Is analytic everywhere on and within 'C'.

   Hence, $\int_c \frac{2z^2 - z - 2}{z-a}dz = 0$ by Cauchy's theorem

2. Evaluate $\int_c (z-a)^n dz$ where C is a simple closed curve and the point $z = a$ is (i) inside C (ii) outside C (n is any integer≠ -1).

   Sol: The parametric equation of C is

   $$z - a = re^{i\theta} \Rightarrow dz = ire^{i\theta}d\theta$$

   And varies from 0 to 2

   (i) $\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta = i\int_0^{2\pi} d\theta$ =(*2 – 0*) = *2*

(ii) If a lies inside C

$$\int_c (z-a)^n dz = \int_0^{2\pi} r^n e^{in\theta} ire^{i\theta} d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - e^0\right]$$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - e^0\right]$$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left[1-1\right] = 0 \qquad \text{if } \neq -1$$

If = *−1* then

$$\int_c (z-a)^n dz = \int_c (z-a)^{-1} dz = \int_c \frac{dz}{z-a} = 2\pi i(by(i))$$

3. Evaluate $\int_c \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3} dz$ with :|| = *2* Using Cauchy's integral formula.

Sol: Given that $f(z) = z^3 - \sin 3z$ is analytic everywhere.

$\therefore f(z)$ Is analytic within :|| = *2*

Here $z = \frac{\pi}{2}$ is a singular point and lies inside c $\quad (\because z = \frac{\pi}{2} \prec 2)$

Since by Cauchy's integral formula, $f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}} dz$

Here = $\frac{\ }{2}$, = *2*

$$\int_c \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3} dz = \frac{2\pi i}{2!} f''(\frac{\pi}{2})$$

$$= \pi i\left[\frac{d^2}{dz^2}(z^3 - \sin 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[\frac{d}{dz}(3z^2 - 3\cos 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[(6z + 9\sin 3z)\right]_{z=\frac{\pi}{2}}$$

$$= \pi i\left[6\frac{\pi}{2}+9\sin 3\frac{\pi}{2})\right]$$

$$= 3\pi i(\pi-3)$$

4. Evaluate $\int_c \frac{dz}{z^3(z+4)}$ where : || = *2* Using Cauchy's integral formula.

Sol: Given that $f(z)=\frac{1}{z^3(z+4)}$

$f(z)$ is not analytic at = *0*, = *−4*

But here = *0* is lies inside 'c' and = *−4* is lies outside' c'

Let $g(z)=\frac{1}{z+4}$ **,** $g(z)$ is analytic on and within C.

By Cauchy's integral formula, we have $g^{(n)}(a)=\frac{n!}{2\pi i}\int_c \frac{g(z)}{(z-a)^{n+1}}dz$

Taking = *0*, = *2*

$$g^{''}(a)=\frac{2!}{2\pi i}\int_c \frac{\frac{1}{z+4}}{(z-0)^{2+1}}dz$$

$$\int_c \frac{dz}{z^3(z+4)} = \pi i g^{''}(0)$$

$$= \pi i\left[\frac{2}{(z+4)^3}\right]_{z=0}$$

$$=\left[\frac{\pi i}{(0+4)^3}\right]=\frac{\pi i}{32}$$

Assignment-Cum-Tutorial Questions

<u>SECTION-A</u>

A. Questions testing the remembering / understanding level of students

I) Multiple Choice Questions

1. Find the real part of $f(z)=z^2$

(a) $x^2 - y^2$ (b) 2xy (c) $x^2 + y^2$ (d) -2xy

2. If w = z^n is analytic, then $\frac{dw}{dz}=$ [ ]

(a) $logz$ (b) nz^n (c)) nz^{n+1} (d)) nz^{n-1}

3. If w = log z, is analytic function, then $\frac{dw}{dz}=$ [ ]

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{z}$ (d) 1

4. If f(z) = z^2, then f(-2+i) = [ ]

(a) $3 + 4i$ (b) $3 - 4i$ (c) $4 - 3i$ (d) $-\pi i$

5. Function which satisfy Laplace's equations in a region R are called ________ in R. [ ]

   (a) Analytic (b) Not Analytic (c) Not harmonic (d) Harmonic

6. The value of K so thet $x^2 + 2x + ky^2$ may be harmonic is [ ]

   (a) 0 (b) 2 (c)1 (d) –1

7. The harmonic conjugate of e^x cosy is [ ]

   (a) $e^x sinx$ (b) $ie^x siny$ (c) $e^x siny$ (d) $e^x cosy$

8. If f(z) = z^3 is [ ]

   (a) Analytic Every Where (b) Not Analytic (c) Not harmonic

   (d) Harmonic

9. The function $f(z) = \bar{z}$ is

   (a) Analytic Every Where (b) Not Analytic (c) Not harmonic

   (d) Harmonic

SECTION-B

I) *Descriptive Questions*

1. Show that both the real and imaginary parts of an analytic function are harmonic.
2. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic.
3. Find all the values of k, such that $f(z) = e^x(cosky + isinky)$ is analytic.
4. Determine P such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.
5. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although Cauchy-Riemann equations are satisfied at that point.
6. Show that the function $f(z) = \frac{x^3(1+i)-y^3(-i)}{x^2+y^2}$ at z ≠ 0 and $f(0) = 0$ is continuous and satisfies Cauchy-Riemann equations at the origin but $f'(0)$ does not exist.
7. If $f(z)$ is analytic, then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$
8. If $f(z)$ is analytic, then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = 2|f'(z)|^2$ Where $f(z)$ is an analytic function.
9. Show that$u(x,y) = x^3 - 3xy^2$ is harmonic.
10. Prove that u= x^2-y^2-2xy-2x+3y is harmonic. Find f(z)= u+iv
11. Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$
12. Find the analytic function$f(z) = u + iv$, if $u - v = e^x$(cosy- siny).

13. If $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$, then show that both $u\ and\ v$ are harmonic but $u + iv$ is not analytic.

C Questions testing the analyzing / evaluating ability of students

1. If $w = \phi + i\psi$ represents the complex potential for an electric field and if the potential function is $\log\sqrt{x^2+y^2}$,find the flux function ?

2. In a two- dimensional flow of fluid, the velocity potential $\phi = x^2 - y^2$ find the stream function ψ

SECTION-C

GATE PREVIOUS QUESTIONS

1. The function w = u+iv $=\frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}$ is not analytic at the point

[GATE 2005]

(a) (0,0) (b) (0,1) (c) (1,0) (d) (2,0)

2. The function w=u+iv $=\frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\frac{y}{x}$ is not analytic at the point

[GATE 2005]

(a) (0,0) (b) (0,1) (c) (1,0) (d) (2,0)

3. An analytic function of a complex variable z = x + iy is expressed as f(z) = u (x, y) + i v(x, y) where i = -1 . If u = xy, the expression for v should be **[GATE-2009]**

(a) $\frac{(x+y)^2}{2}+k$ (b) $\frac{(x-y)^2}{2}+k$ (c) $\frac{\left(y^2-x^2\right)}{2}+k$ (d) $\frac{\left(x^2-y^2\right)}{2}+k$

4. The modulus of the complex number $\frac{(3+4i)}{(1-2i)}$ is. **[GATE-2010]**

(a) 5 (b) $\sqrt{5}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{1}{5}$

5. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u = x^3 - 3x^2y^2$. The expression for v, **[GATE-2010]**

6. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be constant is **[GATE-2011]**

(a) $u = 3y^2 - 3x^2 + k$ (b) $u = 6x - 6y + k$ (c) $u = 6y - 6x + k$ (d) $u = 6xy + k$

7. For an analytic function, $f(x+iy)=u(x,y)+iv(x,y)$,u is given by $u = x^2y^2$. The expression for v, _______________ **[GATE-2014]**

8. For an analytic function, $f(x+iy) = u(x+iy) + v(x+iy)$, u is given by $u = e^{-y}\cos x$. The expression for v, ________________ **[GATE-2014]**

UNIT – VI

Conformal Mapping

Course Objectives:

- To introduce the concepts of conformal and bilinear transformations of standard functions.

Syllabus:
Transformation by e^z, z^2, z^n (n is +ve integer), sin z, cos z, z + a/z translation, rotation, inversion and bilinear transformation- cross ratio – properties – determination of bilinear transformation mapping 3 given points.

Learning Outcomes:
At the end of the unit the student will be able to

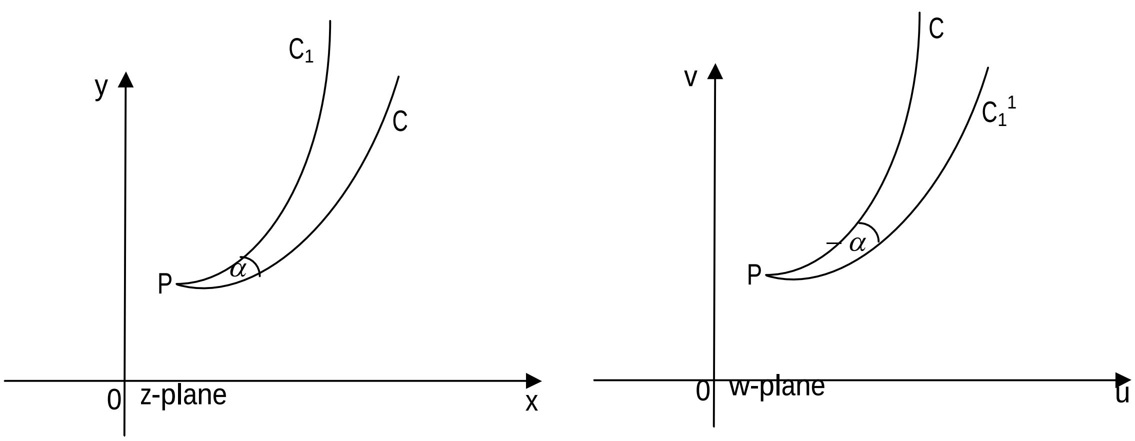
- apply the concepts of conformal and bilinear transformations of standard functions.

Definition: The correspondence defined by equation w = f(z) or u = u (x,y) and v = v (x,y) between the points in the z-plane and w-plane is called a "Mapping" or transformation from z = plane to the w-plane.

The corresponding points w are the w-plane are called the images of the points of z of the z-plane.

Definition: Let there be two curves C and C_1 in the z-plane which intersect at the point p. Let C^1 and C_1^1 be the corresponding curves in the w-plane intersecting at the point p^1.

If the transformation is such that the angle between C and C_1 is equal in magnitude and sense to the angle C^1 and C_1^1 at P_1^1 then the transformation is said to be **conformal**.



The mapping or transformation is said to be isogonal if it preserves the magnitudes of the angles but not sense.

Classification of standard transformation

Translation:- W = Z + C

Where C is any complex constant is a translation, suppose z = x+ iy, c = c_1+ ic_2 and w = u + iv then we get u + iv = (x+iy) + (c_1+ic_2)

Comparing real and imaginary parts we get
$u + iv = (x+iy)+(c_1+ic_2)$
$u = x + c_1 \quad v = y + c_2$
i.e. if p(x,y) is a point in the z-plane then P^1 $(x+C_1, y + C_2)$ is the corresponding point in the z-plane, thus if the w-plane is super imposed on z-plane, the figure is shifted through a distance which is given by C;
Thus this transformation maps a figure in the z-plane to a fig of same shape and size in w-plane.
In particular circles are mapped into circles under this transformation.

Expansion on Contraction and Rotation:-

(Magnification)

Standard Form: w = c z where c is the complex constant.

$Lmt\ Z = re^{i\theta};\ w = \mathrm{Re}^{i\phi},\ C = \beta e^{i\alpha}$

Then w = cz $\Rightarrow \mathrm{Re}^{i\phi} = \beta e^{i\alpha}\ re^{i\theta} = \beta re^{i(\theta+\alpha)}$

We get R = βr , $\phi = \theta + \alpha$

Under this transformation a point p(r,θ) in the z-plane is mapped to the point p^1 $(\beta r, \theta + \alpha)$ in the w-plane. This transformation effects an expansion where |C|>1| and a construction when 0<|c|<1| of this radius vector by $\beta = |c|$and rotation through an angle α = amp (c). Any figure in z-plane is transformed into geometrically a similar figure in the w-plane. In particular circles are mapped to circles.

Inversion and Reflexion:-

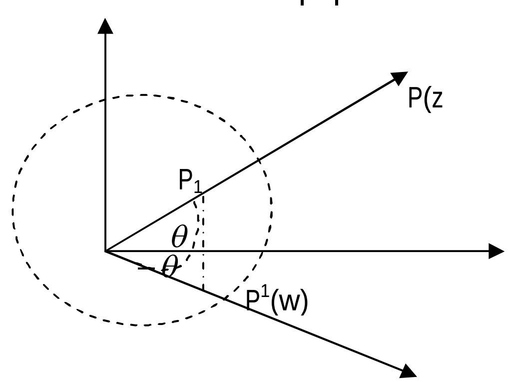
Here it is convenient to think the w-plane as superposed on z-plane. If z = $re^{i\theta}$ and w= $re^{i\varphi}$

then $\mathrm{Re}^{i\phi} = \frac{1}{re^{\theta}} = \frac{1}{r}e^{-i\theta}$

$R = \frac{1}{r} \Rightarrow \phi = -\theta$

When R = $\frac{1}{r}, \theta = -\varphi$

P_1 is the inverse of P w.r.t. when R = $\frac{1}{r},\ and \varphi = -\theta;\ thus\ P\ be(r,\theta)$ and P_1 be $\left(\frac{1}{r},\theta\right)$; P_1 is the inverse of P w.r.t the unit circle with centre 0, then the reflexion of P^1 of P_1 in the real axis represents w = 1/z.
Hence this transformation is an inversion of z writ the unit circle |z|=1 followed by reflexion of the inversion into the real axis.



Bilinear Transformation:-

The transformation $w = \frac{az+b}{cz+d}$ where a,b,c,d are complex constants an d ad-bc$\neq$0 is known as bilinear transformation.

The inverse mapping z = $\frac{-dw+b}{cw-a}$ which is also a bilinear transformation.

Properties:-

1. Invariant points of bilinear transformation; if z maps into itself in the w-plane, w= z then

   $z = \dfrac{az+b}{cz+d}$ or $cz^2 + (d-a)z - b = 0$

   The roots of this equation say z_1, z_2 are invariant or fixed points.
2. If the roots are equal the bilinear transformation is said to be parabolic.
3. A bilinear transformation maps circles into circles.
4. A bilinear transformation preserves cross-ratio of four points.

   $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Problem 1: W = e^z is conformal every where since w is analytic every there

$\dfrac{dw}{dz} = e^z \neq 0$ *for any z*

$\therefore$ e^z has no critical points.

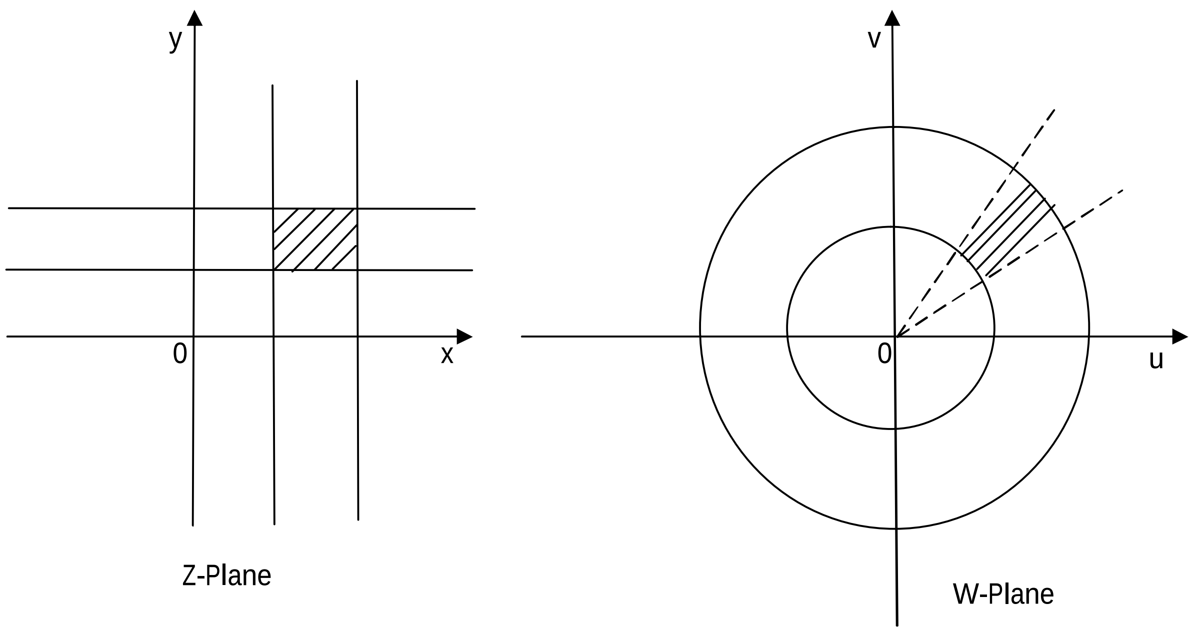
We write z = x + i y and w = $\rho\, e^{i\phi}$.

Hence $\rho e^{i\phi} = e^{x+iy} = e^x e^{iy}$

$\therefore$ $\rho = e^x$ ---------------------------(1)

And $\phi = y$ --------------------------(2)

If x = constant, then from (1) ρ = constant which means lines parallel to y – axis in the z – plane are mapped into circles in the w-plane.



Similarly from (2) if y = constant, then ϕ= constant i.e., the lines parallel to x-axis in the z-plane are mapped into radial lines in the w-plane.

(i) The line y = x

Then from (1) and (2), we have R = e^{ϕ} which represents equiangular spiral in the w plane.

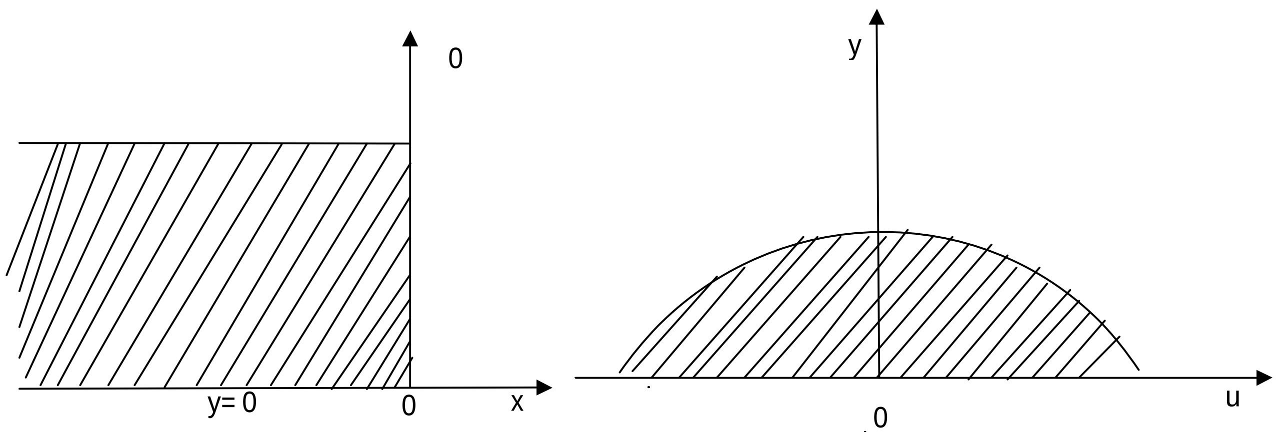
(ii) On y – axis, x = 0

Then from (1), we have

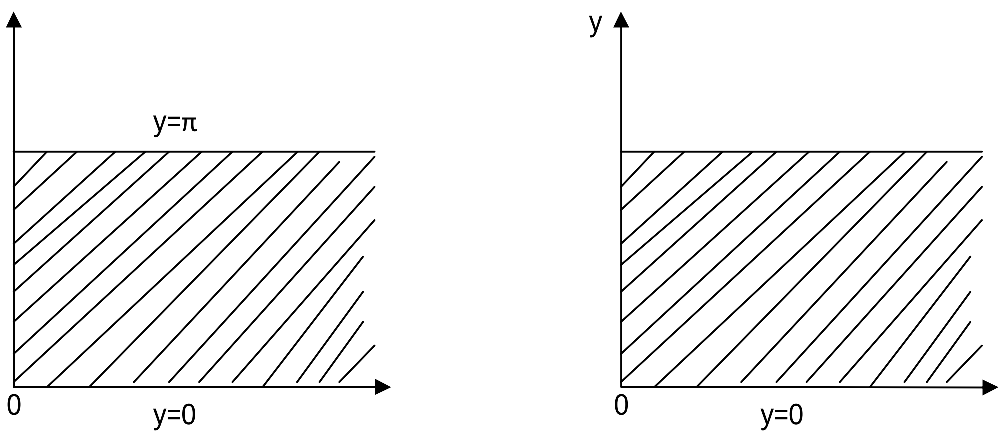
$\rho = e^0 = 1$

Thus the segment of y axis given by $0<y\leq\pi$ in the Z plane corresponds to the unit circle in the w plane.

(iii) The left half of the strip $0 \leq y \leq \pi$ is mapped into the semi circular region given $0<R\leq 1$ and $0\leq\phi\leq\pi$ as shown in the figure.



(iv) The right half of the strip bounded by y = 0 and y = π in the Z plane is mapped onto the upper half plane $I_{mw} > 0$ in the w plane as shown in the figure.



5. The transformation w = sin z:

Then $\frac{dw}{dz} = \cos z = 0 \; when \; z = \frac{\pi}{2} + n\pi, n \in z$

$\therefore$ The mapping w = sin z is conformal at all points except at $z = \frac{\pi}{2} + n\pi$

u + i v = sin (x+iy) = sin x cos iy + cos x sin iy

i.e., u + i v = sin x cosh y + i cos x sinh y

$\therefore$ u = sin x cosh y ………….(1)

and v = cos x sin h y ………….(2)

when x = u = 0 from (1)

and v = sin h y from (2)

If y > 0, sin h y >0 and if y<0, sinh y <0 i.e., the upper half of the imaginary axis (x=0) in the z-plane is mapped into the upper half of the imaginary axis of the w-plane. Similarly the lower half correspond to each other.

Also when y = 0 we get u = sin x from (1) and v = 0 from (2) and since sin x takes the values from -1 to 1, the image of y = 0 is the segment $-1 \leq u \leq 1$.

From (1) and (2),

$$\sin x = \frac{u}{\cosh y} \textit{ and } \cos x = \frac{v}{\sinh y}$$

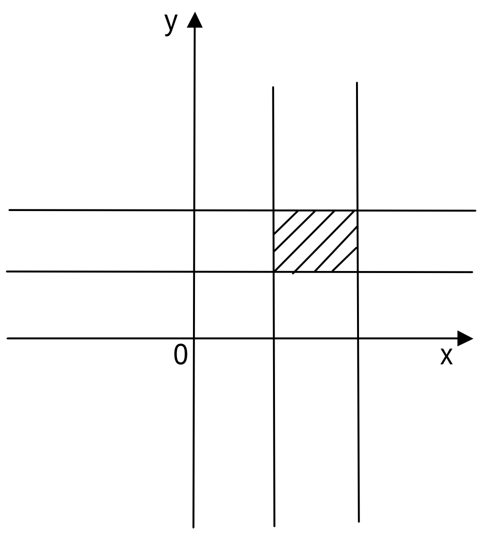
Eliminating x we have

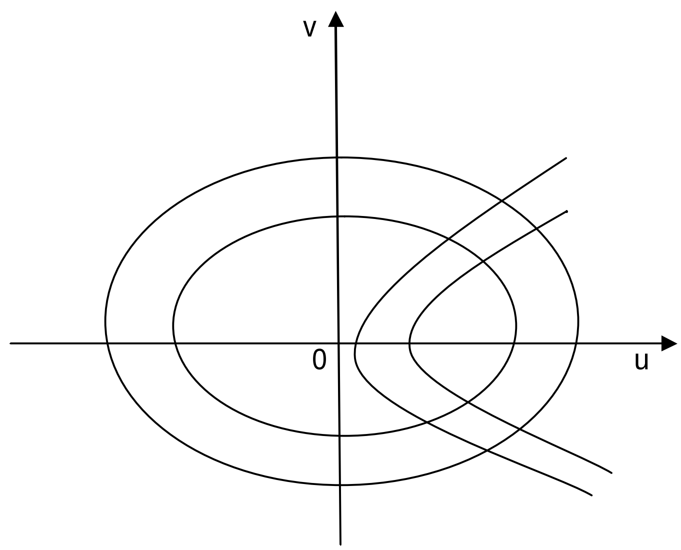
$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

If y = c (constant) $\neq$ 0 then $\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$

i.e., the lines parallel to x-axis are mapped into a family of confocal ellipses in the w-plane. Similarly eliminating y from (1) and (2), we have

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$ If x = k then $$\frac{u^2}{\sin^2 k} - \frac{v^2}{\cos^2 k} = 1$$





Thus the lines parallel to y – axis are mapped into a family of confocal hyperbolas.

6. <u>The transformation w = cosz</u> :

u + i v = cos (x + iy)

= cos x cos iy – sin x sin iy

i.e., u + i v = cos x cosh y – i sin x sinh y

$\therefore$u = cos x coshy …………………(1)

And v = - sin x sin h y …………………(2)

Eliminating y from (1) and (2), we have

$$\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

If x = c (constant) then we have $\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 1$

Thus the lines parallel to the y-axis are mapped into confocal hyperbolas in the w-plane
Similarly eliminating x from equations (1) and (2), we have

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

If y = k (constant) then $\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$

Thus straight lines parallel to x-axis are transformed into a family of ellipses.

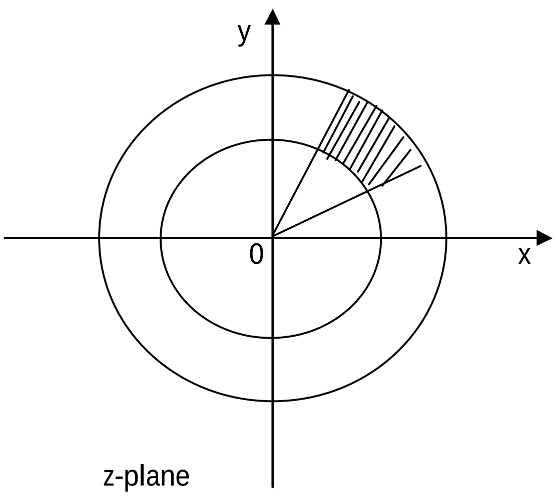
7.The transformation w = $\log_e z$:

Let w = u + iv and z = r $e^{i\theta}$

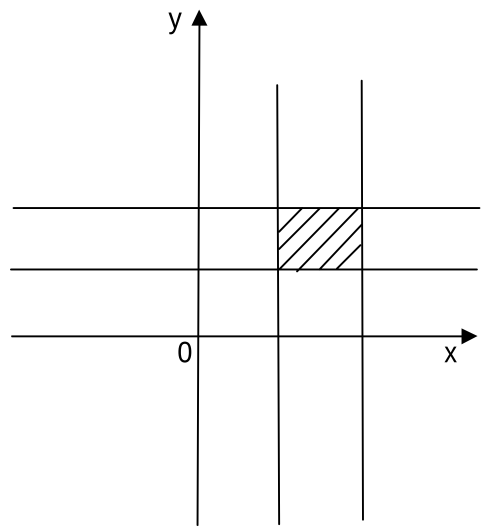
$\therefore$ u + i v = $\log_e r e^{i\theta}$ = log r + $\log_e e^{i\theta}$ = log r + i θ

Hence u = log r

And v = θ



z-plane



w-plane

From (1) if r = c (constant) u = log c (constant)
i.e., the circles in the z-plane are mapped into straight lines parallel to the v – axis in the w plane.
Similarly if θ = k (constant), then v = k from (2) i.e., the radial line in the z-plane are mapped into a family of straight lines parealle to the u – axis in the w-plane.
From the given transformation we obtain

$$\frac{dw}{dz} = \frac{1}{2}$$

The derivation is infinite at z = 0
$\therefore$ The mapping is conformal except at z = 0

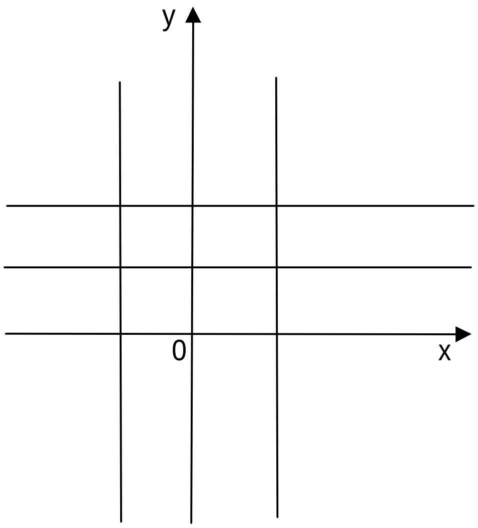
8. The transformation w = cosh z:

The $\frac{dw}{dz} = \sinh z = 0 \;\; if \; z' = 0, \;\; \neq \pi i, 2\pi i ......$

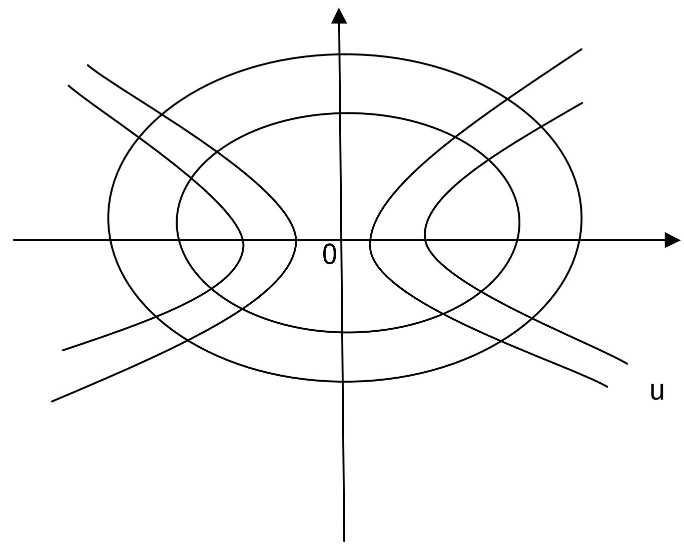
$\therefore$ The mapping is conformal at all points except at z = n π i, n $\in$ z
u + i v = cosh (x+iy)
= cosh x cosh i y + sinh x sinh iy
$\therefore$ u + i v = cosh x cos y + i sinh x siny
Hence u = cosh x cosy ………………(1)
And v = sinh x sin y ………………(2)
Eliminating x from (1) and (2), we get

$$\frac{u^2}{\cos^2 y}-\frac{v^2}{\sin^2 y}=1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Which show s that the lines parallel to x-axis (i.e., y = constant) in the z-plane are mapped into hyperbolas in the w-plane.



z-plane



w -plane

Problems:

1. Find the bilinear transformation that maps the points –i, 0 , i into the points -1, i, 1 respectively.

   Sol:

   Let the transformation be

   $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

   $$\frac{(w+1)(i-1)}{(-1-i)(1-w)}=\frac{(z+i)(0-i)}{(-i-0)(i-z)}$$

   $$\therefore \frac{(w+1)(1-i)}{(w-1)(1+i)}=\frac{(z+i)}{(z-i)}$$

   Applying Componendo – Dividendo principle

   $$\frac{(w+1)(1-i)+(w-1)(1+i)}{(w+1)(1-i)-(w-1)(1+i)}=\frac{z+i+z-i}{z+i-z+i}$$

   $$\frac{w-iw+1-i+w+iw-1-i}{w-iw+1-i-w-iw+1+i}=\frac{2z}{2i}$$

   $$\frac{2(w-i)}{2(1-iw)}=\frac{z}{i},\ \frac{w-i}{1-iw}=\frac{z}{i}$$

   i(w-i)=z(1-iw)

   iw+1 = z – iwz

   $$\therefore iw(z+1)=z-1$$

   $$\therefore w=\frac{z-1}{i(z+1)} \quad \therefore w=-\frac{(1-z)}{i(1+z)}$$

i.e., $w = \frac{i(1-z)}{(1+z)}$

2. Find a bilinear transformation which maps the points -1, 0, 1 into the points -0, -1, ∞ respectively.

Sol:

Let the transformation be

w = $\frac{az+b}{cz+d}$ -----------------------(1)

when z = -1

w = 0 $\therefore 0 = \frac{-a+b}{-c+d}$

$\Rightarrow$ - a + b = 0 -----------------------(2)

When z = 0

w = -1 $\therefore -1 = \frac{b}{d}$

$\Rightarrow b+d=0$ ----------------------(3)

When z = 1

W = ∞ $\therefore \infty = \frac{a+b}{c+d}$

$\Rightarrow c+d=0$ ----------------------(4)

from (2) b = a

from (3) d = -b = -a

from (4) c = -d =a

Substituting the values b, c, d in (1)

$w = \frac{az+a}{az-a}$ $\therefore w = \frac{z+1}{z-1}$s

3. Find the bilinear transformation that maps the points $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ into the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$.

Sol:

Let the transformation be

$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$ --(1)

Since $z_1 = \infty$

Put $z_1 = \frac{1}{z_1'}$ since $w_3 = \infty$

Put $w_3 = \frac{1}{w_3'}$

from (1) we get

$\frac{(w-w_1)(w_2w_3-1)}{(w_1-w_2)(1-ww_3')} = \frac{(zz_1'-1)(z_2-z_3)}{(1-z_1'z_2)(z_3-z)}$

Since $z_1 = \infty$

$z_1' = 0$

Again since $w_3 = \infty$ $\therefore w_3' = 0$

$$\therefore \frac{(w-0)(i.0-1)}{(w_1-w_2)(1-ww_3')} = \frac{(z.0-1)(i-0)}{(1-0.i)(0-z)}$$

$$\frac{w(-1)}{-i.1} = \frac{-i}{-z} \qquad \therefore \frac{w}{i} = \frac{i}{z}$$

$$\Rightarrow w = \frac{i^2}{z} \qquad i.e., \ w = -\frac{1}{z}$$

4. Determine the bilinear transformation that maps the points 1 -2i, 2 + i, 2 + 3i into the points 2 + i, 1 + 3i, 4.

Sol:

Let $z_1 = 1 - 2i$, $z_2 = 2 + i$, $z_3 = 2 + 3i$

$w_1 = 2 + i$, $w_2 = 1 + 3i$, $w_3 = 4$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(1-2i-2-i)(2+3i-z)}{(1-2i-z)(2+3i-2-i)} = \frac{(2+i-1-3i)(4-w)}{(2+i-w)(4-i-3i)}$$

$$\left(\frac{2+3i-z}{1-2i-z}\right) = \frac{1}{3}\frac{(w-4)}{(2+i-w)}$$

w(4z-7-7i) = -7 – 16i + z (10+3i)

$$w = \frac{z(10+3i)-7-16i}{4z-7-7i}$$

5. Find the bilinear transformation that maps the points, $(\infty, i, 0)$ into the points $(0,1,\infty)$

Sol:

Let $z_1 = \infty$, $z_2 = i$, $z_3 = 0$, $w_1 = 0$, $w_2 = i$, $w_3 = \infty$

Substituting these values in

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

We get $\dfrac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \dfrac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$

Or $\dfrac{1}{1}w = \dfrac{i}{z}$ $i.e., \ w = \dfrac{i}{z}$

Which is the required transformation

Since $z_1 = \infty$, $\qquad z_1 = \dfrac{1}{z_1'}$

Since $w_3 = \infty$

Put $w_3 = \dfrac{1}{w_3'}$

From (1) we get

$$\frac{(w-w_1)(w_2w_3'-1)}{(w_1-w_2)(1-ww_3')} = \frac{(zz_1'-1)(z_2-z_3)}{(1-z_1'z_2)(z_3-z)}$$

Since $z_1 = \infty$ $\quad z_1' = 0$

Since $w_3 = \infty$ $\quad \therefore w_3' = 0$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(0-1)(1-0)}{(1-0)(0-z)}$$

6. Find the bilinear transformation which maps the points (1, i, -1) into the points $(0,1,\infty)$.

Sol:

Let $z_1 = 1$, $\quad z_2 = i$, $\quad z_3 = -1$

$w_1 = 0$, $\quad w_2 = 1$, $\quad w_3 = \infty$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

From (1) we get

$$\frac{(w-w_1)(w_2w_3-1)}{(w_1-w_2)(1-ww_3')} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Again since $w_3 = \infty$ $\quad \therefore w_3' = 0$

$$\Rightarrow \frac{(w-0)(1\times 0-1)}{(0-1)(1-w.0)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \frac{-w}{-1} = \frac{(z-1)(i+1)}{-(1-i)(1+z)}$$

$$w = -\left(\frac{z-1}{z+1}\right)\frac{1+i}{1-i}\times\frac{1+i}{1+i}$$

$$w = -\left(\frac{z-1}{z+1}\right)\left(\frac{(1+i)^2}{1+1}\right) = -\left(\frac{z-1}{z+1}\right)\left(\frac{1+i^2+2i}{2}\right)$$

$$= -\left(\frac{z-1}{z+1}\right)\left(\frac{2i}{2}\right) = -i\left(\frac{z-1}{z+1}\right) = \frac{i(1-z)}{1+z}$$

7. Plot the image of $1<|z|<2$ under the transformation w = 2i z +1.

Sol:

w = 2 i z + 1

$\Rightarrow u+iv = 2i(x+iy)+1$

$\therefore u = -2y+1$

v= 2x

i.e., $x = \frac{v}{2}$ $\quad and \quad$ $y = \frac{1-u}{2}$

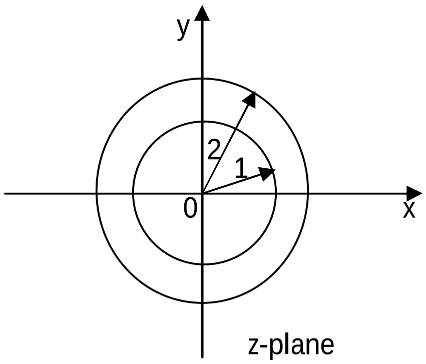
now $\quad |z| = 1$

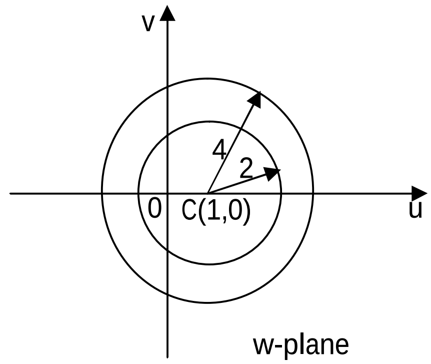
$\Rightarrow x^2+y^2 = 1$ $\qquad$ substituting for x and y

$$\frac{v^2}{4}+\frac{(1-u)^2}{4} = 1$$

$$\Rightarrow (u-1)^2+v^2 = 4$$

i.e., the circle |z| = 1 in the z-plane is mapped to the circle $(u-1)^2 + v^2 = 4$ in the w plane.





Similarly |z| = 2 $\Rightarrow x^2 + y^2 = 4$. On substituting for x and y we get $\frac{v^2}{4} + \frac{(1-u)^2}{4} = 4 \Rightarrow (u-1)^2 + v^2 = 16$ (ii) which is again a circle in the w-plane with centre (1,0) and radius = 4. Hence the region between the two concentric circle |z| = 1 and |z| = 2 is mapped on to the annular region between the two concentric circles given by (i) and (ii) in the w-plane.

8. Find the image of the following curves under the mapping $w = \frac{1}{z}$ (i) the line y – x + 1 = 0 (ii) the circle |z-3| = 5.

Sol:

$$w = \frac{1}{z} \quad or \quad z = \frac{1}{w}$$

i.e., $x + iy = \frac{1}{u+iv} = \frac{u-iv}{(u-iv)(u+iv)} = \frac{u-iv}{u^2+v^2}$

$$\therefore x = \frac{u}{u^2+v^2}$$

$$y = -\frac{v}{u^2+v^2}$$

(i) Hence the line y – x + 1 = 0 is mapped to

$$-\frac{v}{u^2+v^2} - \frac{u}{u^2+v^2} + 1 = 0$$

i.e., $u + v = u^2 + v^2$ in the w plane

(ii) |z-3|=5

$$\Rightarrow \left|\frac{1}{w} - 3\right| = 5$$

i.e., $\left|\frac{1-3w}{w}\right| = 5 \quad or \left|\frac{1-3(u+iv)}{u+iv}\right| = 5$

$$\Rightarrow \frac{|(1-3u) - 3iv|}{|u+iv|} = 5$$

or $\frac{\sqrt{(1-3u)^2+9v^2}}{\sqrt{u^2+v^2}}=5$ squaring and cross multiplying

$(1-3u)^2+9v^2=25(u^2+v^2)$

$\Rightarrow u^2+v^2+\frac{3}{8}u=\frac{1}{16}$

Or $\left(u+\frac{3}{16}\right)^2+v^2=\frac{25}{256}$ *or* $\left|w+\frac{3}{16}\right|=\frac{5}{16}$

9. Show that function $w=\frac{4}{z}$ transforms the straight line x = c in the z plane into a circle in the w plane.

Sol:

$w=\frac{4}{z}$

$i.e., \ u+iv=\frac{4}{x+iy}=\frac{4(x-iy)}{(x-iy)(x+iy)}$

$\therefore \ u+iv=\frac{4(x-iy)}{x^2+y^2}$

$\Rightarrow u=\frac{4x}{x^2+y^2}$

$v=\frac{-4y}{x^2+y^2}$

When x = c

$u=\frac{4c}{c^2+y^2}$

and $v=\frac{-4y}{c^2+y^2}$

From (i)

$c^2+y^2=\frac{4c}{u}$ or $y^2=\frac{4c}{u}-c^2=\frac{4c-c^2u}{u}$

Squaring (ii)

$v^2=\frac{16y^2}{(c^2+y^2)^2}$ substituting for y^2 and c^2+y^2

$\therefore v^2=\frac{16(4c-c^2)}{u}\times\frac{u^2}{16c^2}$

i.e., $v^2=\frac{(4-cu)u}{c}$

$\Rightarrow cv^2=(4-cu)u$

Or $u^2+v^2-\frac{4}{c}u=0$ which is a circle in the w-plane

10. Under the transformation w = $\frac{1}{z}$, find the image of the circle |z-2i| = 2.

Sol: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$, $x + iy = \frac{1}{u + iv}$

$$x + iy = \frac{u - iv}{(u + iv)(u - iv)}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

$$x + iy = \frac{u}{u^2 + v^2} - \frac{iv}{u^2 + v^2}$$

$$\therefore \quad x = \frac{u}{u^2 + v^2}, \; y = \frac{-v}{u^2 + v^2}$$

Now |z-2i|= 2

|x + iy – 2i| = 2

$|x+i(y-2)|^2 = 4$

$x^2 + (y - 2)^2 = 4$

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + y^2 - 4y = 0$

Substituting for x and y, we get

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - 4\left(\frac{-v}{u^2 + v^2}\right) = 0$$

$$\Rightarrow \frac{u^2 + v^2 + 4v(u^2 + v^2)}{(u^2 + v^2)^2} = 0$$

$$u^2 + v^2 + 4v(u^2 + v^2) = 0$$

$$(u^2 + v^2)[1 + 4v] = 0$$

$\therefore$1+4v = 0

i.e., |z-2i| = 2 is a circle in z-plane is mapped to the line 1+ 4v = 0 in the w-plane.

11. Show that the transformation $w = \frac{1}{z}$ maps a circle to a circle a straight line if the former goes through the origin.

Sol:

$$\omega = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2} \quad \Rightarrow x = \frac{u}{u^2 + v^2}$$

General form of circle equation is

$$a(x^2 + y^2) + bx + cyd = 0 (a \neq 0)$$

$$a\left(\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right)+b\left(\frac{u}{u^2+v^2}\right)+c\left(\frac{-v}{u^2+v^2}\right)+d=0$$

$$\Rightarrow \frac{a}{u^2+v^2}+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}+d=0$$

$$\Rightarrow a+bu-cv+d(u^2+v^2)=0$$

$$\Rightarrow d(u^2+v^2)+bu-cv=a=0$$

Which again represents a circle equation.

$\omega=\frac{1}{z}$ maps a circle to a circle.

if the former goes through origin.

i.e., equation of circle passing through origin i.e., d = 0 is

$$a(x^2+y^2)+bx+cy=0$$

$$a\left(\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right)+b\left(\frac{u}{u^2+v^2}\right)+c\left(\frac{-v}{(u^2+v^2)^2}\right)=0$$

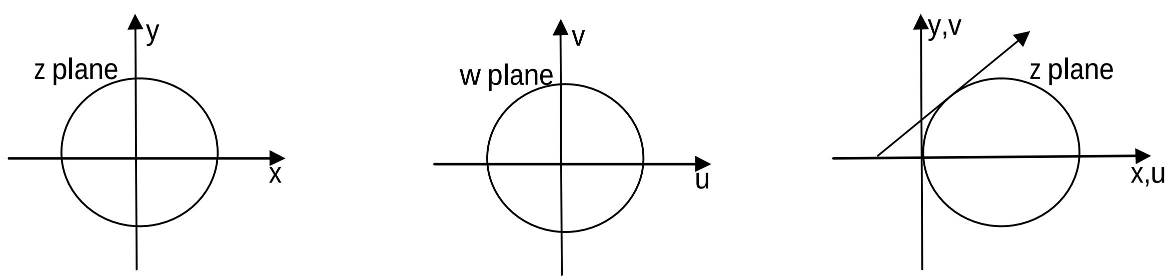
$$\Rightarrow \frac{a}{u^2+v^2}+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}=0$$

$$\Rightarrow a+bu-cv=0$$

$$\Rightarrow bu-cv+a=0$$

Which represents a straight line equation in u and v.

$\therefore \omega=\frac{1}{z}$ maps a circle to a straight line if it passes through origin.



12. Find the bilinear transformation which maps the points z = 1, i, -1 onto the points w = I, 0, -i.

Hence find (a) the image of |z| <1,

(b) the invariant points of this transformation.

Sol:

Let the points z_1 = 1, z_2 =I, z_3 = -1 and z_4 = z map onto the points w_1 = I, w_2 = 0, w_3 = -i, and w_4 = w.

Since the cross-ration remains unchanged under a bilinear transformation.

$$\therefore \frac{(1-i)(-1-z)}{(1-z)(-1-i)}=\frac{(i-0)(-i-w)}{(i-w)(-i-0)}$$

Or $$\frac{w+i}{w-i}=\frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\frac{2w}{2i} = \frac{(z+1)(1-i)(z-1)(1+i)}{(z+1)(1-i)-(z-1)(1+i)}$

$w = \frac{1+iz}{-iz}$ ....(i)

Which is the required bilinear transformation.

(a) Rewriting (i) as z = $i\frac{1-w}{1+w}$

$\therefore \left|\frac{i(1-w)}{1+w}\right| = |z| < 1 \;\; or \;\; |i|\,|1-w| < |1+w|$

Or |1-u-iv| < |1+u+iv| [$\because$ |i|=1]

Or $(1-u)^2 + v^2 < (1+u)^2 + v^2$ which reduces to u > 0.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(b) To find the invariant points of the transformation, we put w = z in (i).

$\therefore \; z = \frac{1+iz}{1-iz} \;\; or \;\; iz^2 + (i-1)z + 1 = 0$

Or $z = \frac{1-i \pm \left\lfloor (i-1)^2 \right\rfloor}{2i} = -\frac{1}{2}\{1+i \pm (6i)\}$

Which are the required invariant points.

Assignment-Cum-Tutorial Questions:

Section-A

1. Bilinear linear transformation is used for transforming an analog filter to a digital filter.

   (a) True (b) False

2. What is the image of the infinite strip bounded by x = 0 and x = pi/4 under the transformation w = cosz

   (a) half hyperbola (b) hyperbola (c) parabola (d) ellipse

3. What is the image of the circle with centre ‘a’ and radius ‘c’ under the transformation w = z^2

   (a) cardoid (b) limacon (c) lemniscates (d) circles

4. What are the fixed points of the transformation $w = \frac{z-1}{z+1}$

   (a) 1, -1 (b) 1, i (c) i, -i (d) i, 1

5. The transformation $w = z^{1/\alpha}$ maps

   (a) Half- planes into circles (b) Sectors into half-planes

   (c) Half-planes into confocal hyperbolas (d) none of these

6. If the mapping w = f(z) is conformal then the function f(z) is

(a) analytic (b) non-analytic (c) harmonic (d) none of the these

7. Inverse transformation w = 1/z transforms the straight line ay + bx = 0 into

   (a) circle (b) straight line (c) parabola (d) ellipse

8. Bilinear transformation always transforms circles into

   (a) circles (b) parabolas (c) hyperbolas (d) ellipses

SECTION-B

1. Find the image of the triangle with vertices at i, 1+ i, 1-i in the z-plane under the transformation w = 3z + 4 – 2i.
2. Find the image of the circle |z| =2 under the transformation w = z + 3 +2i.
3. Find the image of the following curves under the mapping $w = \frac{1}{z}$ (i) the line y – x +1 = 0 (ii) the circle |z-3| = 5.
4. Under the transformation $w = \frac{1}{z}$, find the image of the circle |z-2i|=2.
5. Show that the transformation $w = \frac{1}{z}$ maps a circle to a circle a straight line if the former goes through the origin.
6. Find the image of |z| = 2 under the transformation $\omega = 3z$
7. Find the bilinear transformation that maps the points –i, 0, i into the points -1, i, 1 respectively.
8. Find a bilinear transformation which maps the points -1, 0, 1 into the points -0, -1, ∞ respectively.
9. Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ into the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$.
10. Determine the bilinear transformation that maps the points 1 – 2i, 2+i, 2+3i into the points 2+i, 1+3i, 4.

***